International Journal of Contemporary Mathematical Sciences
Vol. 9, 2014, no. 12, 569 - 578
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2014.4885

k-Fractional Trigonometric Functions

Rubén A. Cerutti and Luciano L. Luque

Faculty of Exact Sciences
National University of the Northeast
Av. Libertad 5540 (3400); Corrientes, Argentina

Abstract

Based on the k-Mittag-Lefler function and the k-α-Exponential Function we introduce families of functions that allows us define new fractional trigonometric functions that contain the classical trigonometric functions as particular case for some convenient election of parameters. We study some elementary properties and obtain the Laplace transform of some elements of the families.

Keywords: Fractional Calculus, k-Calculus, Exponential Function

I Introduction

As is well known the classical exponential function, and from it, the classical trigonometric functions play an important role in the solution of ordinary differential equation with constant coefficients. Hence the importance of studying it and its generalizations including the Mittag-Leffler function of one or two parameters, the one introduced by Prajapati (c.f. [8]) or for us in [5] the called k-Mittag-Leffler function given by

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ z^n}{\Gamma_k(n\alpha+\beta)n!} \tag{I.1}$$

where $(\gamma)_{n,k}$ denote the k-Pochhammer symbol given by

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+(n-1)k) \tag{I.2}$$

and $\Gamma_k(z)$ is the k-Gamma Function due to Diaz and Pariguan (c.f.[4])

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt. \tag{I.3}$$

Elementary calculations allow us to establish the following relationship between $(\gamma)_{n,k}$ and $\Gamma_k(z)$

$$(\gamma)_{n,k} = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}. \tag{I.4}$$

It can be seen that if in (I.1), is taken $k = \gamma = \alpha = \beta = 1$, the classical exponential function is obtained

$$E^1_{1,1,1}(z) = e^z. \tag{I.5}$$

By Euler equation can be established that

$$e^z = e^{x+iy} = e^x \left(\cos y + i \sin y\right), \tag{I.6}$$

and considering the series expansion of the exponential function and the sine and cosine functions we have

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \tag{I.7}$$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}. \tag{I.8}$$

The extension to fractional calculus of the exponential function carries with it the extension of trigonometric functions. It may be seen the elementary presentation (c.f.[1]) and the very interesting one given in [3] in which starting from a generalized exponential function

$$\begin{aligned} e_\alpha^{\lambda(x-a)} &= (x-a)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\lambda^n (x-a)^{n\alpha}}{\Gamma\left[\alpha(n+1)\right]} \\ &= (x-a)^{\alpha-1} E_{\alpha,\alpha}\left[\lambda(x-a)^\alpha\right], \end{aligned} \tag{I.9}$$

where $x > a$, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$ and $E_{\alpha,\beta}(x)$ is the two parametters Mittag-Leffler function.

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+\beta)} \tag{I.10}$$

may be extended to Fractional Calculus definitions of sine and cosine function by

$$\cos_\alpha\left[\lambda(x-\alpha)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n}(x-a)^{(2n+1)\alpha-1}}{\Gamma\left[(2n+1)\alpha\right]} \tag{I.11}$$

and

$$\sin_\alpha\left[\lambda(x-\alpha)\right]=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n+1}(x-a)^{(n+1)2\alpha-1}}{\Gamma\left[(n+1)2\alpha\right]} \tag{I.12}$$

verifying

$$D^{\alpha}_{a^+}\left[\sin_\alpha(\lambda x)\right]=\lambda\cos_\alpha(\lambda x) \tag{I.13}$$

and

$$D^{\alpha}_{a^+}\left[\cos_\alpha(\lambda x)\right]=-\lambda\sin_\alpha(\lambda x) \tag{I.14}$$

where $D^{\alpha}_{a^+}$ denote the Riemann-Liouville fractional derivative of order α defined by

$$\left(D^{\alpha}_{a^+}f\right)(x)=\left[D^nI^{n-\alpha}_{a^+}f\right](x) \tag{I.15}$$

being D the usual derivative operator and $I^{\alpha}_{a^+}$ the fractional Riemann-Liouville integral operator of order α.

Through this paper we will use frequently

$$D^{\gamma}\left[(x-a)^{\alpha}\right]=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)}(x-a)^{\alpha-\gamma} \tag{I.16}$$

(c.f. [2]), and

$$\Gamma(x)=\left(\frac{1}{k}\right)^{x-1}\Gamma_k(kx) \tag{I.17}$$

(c.f.[4]).

II Riemann-Liouville Fractional Derivative of order $\frac{a}{k}$ of the k-α-Exponential Function

In this paragraph, based on the k-α-Exponential Function, new definitions of trigonometric functions are presented and some elementary properties of them are studied. To do that we will begin computing the Riemann-Liouville fractional derivative of order α/k of the k-α-Exponential Function.

Let ${}_ke^{az}_{\gamma,\alpha}$ be the k-α-Exponential Function (c.f.[7]) given by

$$\begin{aligned}{}_ke^{az}_{\gamma,\alpha} &= z^{\frac{\alpha}{k}-1}E^{\gamma}_{k,\alpha,\alpha}\left(az^{\frac{\alpha}{k}}\right)\\ &= z^{\frac{\alpha}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n,\gamma}\left(az^{\frac{\alpha}{k}}\right)^n}{\Gamma_k\left[(n+1)\alpha\right]n!}.\end{aligned} \tag{II.1}$$

Taking into account formulae (I.16) we have

$$D^{\frac{\alpha}{k}}\left[{}_k e^{az}_{\gamma,\alpha}\right]=\sum_{n=1}^{\infty}\frac{(\gamma)_{n,k}a^n\Gamma\left[\frac{\alpha}{k}(n+1)\right]z^{\frac{\alpha}{k}n-1}}{\Gamma_k\left[(n+1)\alpha\right]n!\;\Gamma\left[\frac{\alpha}{k}(n+1)-\frac{\alpha}{k}\right]}. \tag{II.2}$$

Reminding the relationship between the two Gamma Functions given by (I.17) we have

$$\begin{aligned} D^{\frac{\alpha}{k}}\left[{}_k e^{az}_{\gamma,\alpha}\right] &= \sum_{n=1}^{\infty}\frac{(\gamma)_{n,k}a^n k^{1-\frac{\alpha}{k}(n+1)}\Gamma_k\left[\alpha(n+1)\right]z^{\frac{\alpha}{k}n-1}}{\Gamma_k\left[\alpha(n+1)\right]n!\;k^{1-\left[\frac{\alpha}{k}(n+1)-\frac{\alpha}{k}\right]}\Gamma_k\left[\alpha(n+1)-\alpha\right]} \\ &= \frac{az^{\frac{\alpha}{k}-1}}{k^{\frac{\alpha}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{n+1,k}\left(az^{\frac{\alpha}{k}}\right)}{(n+1)!\;\Gamma_k\left[\alpha(n+1)\right]}. \end{aligned} \tag{II.3}$$

If in (II.3); $k=\gamma=1$ is considered we obtain the formulae (23) of [2].

When $a=i\lambda$, $\lambda\in\mathbb{R}$, and $x-a=t>0$, (II.1) can be written

$$\begin{aligned} {}_k e^{i\lambda t}_{\gamma,\alpha} &= t_k^{\frac{\alpha}{k}-1}E^{\gamma}_{\alpha,\alpha}\left(i\lambda t^{\frac{\alpha}{k}}\right) \\ &= \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}i^n\lambda^n t^{\frac{\alpha}{k}(n+1)-1}}{\Gamma_k\left[(n+1)\alpha\right]n!} \end{aligned} \tag{II.4}$$

where i is the imaginary unit.

By grouping summands according to the powers of the imaginary unit and the parity of n, we obtain

$${}_k e^{i\lambda t}_{\gamma,\alpha}=\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2n,k}\lambda^{2n}t^{\frac{\alpha}{k}(2n+1)-1}}{\Gamma_k\left[(2n+1)\alpha\right](2n)!}+i\sum_{n=0}^{\infty}\frac{(-1)^n(\gamma)_{2n+1,k}\lambda^{2n+1}t^{2\frac{\alpha}{k}(n+1)-1}}{\Gamma_k\left[2\alpha(n+1)\right](2n+1)!}. \tag{II.5}$$

Taking into account the above, we define

$${}_k\cos_{\gamma,\alpha}(\lambda t)=\mathrm{Re}\left({}_k e^{i\lambda t}_{\gamma,\alpha}\right) \tag{II.6}$$

and

$${}_k\sin_{\gamma,\alpha}(\lambda t)=\mathrm{Im}\left({}_k e^{i\lambda t}_{\gamma,\alpha}\right). \tag{II.7}$$

If $\gamma=k=1$, coincides with formulaes (20) and (21) from [3]. It may be observed that if we adopt $k=\gamma=1$, we have

$${}_1\cos_{1,\alpha}(\lambda t)=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n}t^{k(2n+1)-1}}{\Gamma_k\left[(2n+1)\alpha\right]}$$

and

$${}_1\sin_{1,\alpha}(\lambda t)=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n+1}t^{k(2n+2)-1}}{\Gamma_k\left[(2n+2)\alpha\right]}$$

which coincides when $\lambda = 1$ with the ones in [2]. Then, it may be written

$$ {}_ke^{i\lambda t}_{\gamma,\alpha} = {}_k\cos_{\gamma,\alpha}(\lambda t) + i\ {}_k\sin_{\gamma,\alpha}(\lambda t). \tag{II.8}$$

From (II.6) and (II.7) when $k = \alpha = \gamma = 1$ we have

$$ {}_1\cos_{1,1}(t) = \cos(t) = \mathrm{Re}\left(e^{it}\right)$$

and

$$ {}_1\sin_{1,1}(t) = \sin(t) = \mathrm{Im}\left(e^{it}\right).$$

Analogously, it may be obtained

$$ {}_ke^{-i\lambda t}_{\gamma,\alpha} = {}_k\cos_{\gamma,\alpha}(\lambda t) - i\ {}_k\sin_{\gamma,\alpha}(\lambda t). \tag{II.9}$$

From (II.8) and (II.9) it result

$$ {}_k\cos_{\gamma,\alpha}(\lambda t) = \frac{{}_ke^{i\lambda t}_{\gamma,\alpha} + {}_ke^{-i\lambda t}_{\gamma,\alpha}}{2} \tag{II.10}$$

and

$$ {}_k\sin_{\gamma,\alpha}(\lambda t) = \frac{{}_ke^{i\lambda t}_{\gamma,\alpha} - {}_ke^{-i\lambda t}_{\gamma,\alpha}}{2i}. \tag{II.11}$$

Can be clearly seen that when $k = \alpha = \gamma = 1$ results the known expression for the classical trigonometric functions given by (I.7) and (I.8).

From the defining formulae (II.10) and (II.11) may be derived the following properties:

$$ {}_k\cos_{\gamma,\alpha}(-\lambda t) = {}_k\cos_{\gamma,\alpha}(\lambda t) \tag{II.12}$$

$$ {}_k\sin_{\gamma,\alpha}(-\lambda t) = -{}_k\sin_{\gamma,\alpha}(\lambda t) \tag{II.13}$$

$$ {}_k\cos^2_{\gamma,\alpha}(\lambda t) + {}_k\sin^2_{\gamma,\alpha}(\lambda t) = {}_k\ e^{i\lambda t}_{\gamma,\alpha}\ {}_ke^{-i\lambda t}_{\gamma,\alpha} \tag{II.14}$$

By replacing k,α, γ and λ for 1 in (II.14) is obtained the well known relation

$$ \cos^2(t) + \sin^2(t) = 1. \tag{II.15}$$

III The Laplace Transform of the k-α-Exponential Function

We begin by recalling the Laplace Transform of the function $t^{\frac{\alpha}{k}(n+1)-1}$, given by

$$ \mathcal{L}\left[t^{\frac{\alpha}{k}(n+1)-1}\right](s) = \frac{\Gamma\left[\frac{\alpha}{k}(n+1)\right]}{s^{\frac{\alpha}{k}((n+1))}} \tag{III.1}$$

(c.f.[7]).

The series expansion of the k-α-Exponential Function allows us evaluate

$$\begin{aligned}\mathcal{L}\left[{}_k e^{\lambda t}_{\gamma,\alpha}\right] &= \mathcal{L}\left[\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ \lambda^n t^{\frac{\alpha}{k}(n+1)-1}}{\Gamma_k\left[(n+1)\alpha\right] n!}\right] \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ \lambda^n \mathcal{L}\left[t^{\frac{\alpha}{k}(n+1)-1}\right]}{\Gamma_k\left[(n+1)\alpha\right] n!},\end{aligned}$$

and taking into account (III.1), it results

$$\begin{aligned}\mathcal{L}\left[{}_k e^{\lambda t}_{\gamma,\alpha}\right] &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ \lambda^n \Gamma\left[\frac{\alpha}{k}(n+1)\right]}{n!\Gamma_k\left[(n+1)\alpha\right] s^{\frac{\alpha}{k}(n+1)}} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ k^{1-\frac{\alpha}{k}n-\frac{\alpha}{k}} a^n}{n! s^{\frac{\alpha}{k}n} s^{\frac{\alpha}{k}}} \\ &= \frac{k^{1-\frac{\alpha}{k}}}{s^{\frac{\alpha}{k}}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{n!} \left(\frac{\lambda}{(ks)^{\frac{\alpha}{k}}}\right)^n. \qquad \text{(III.2)}\end{aligned}$$

According the notation by [8] formula (2.22) we may writte

$$\mathcal{L}\left[{}_k e^{\lambda t}_{\gamma,\alpha}\right] = \frac{k^{1-\frac{\alpha}{k}}}{s^{\frac{\alpha}{k}}} \left(1 - \frac{\lambda}{(ks)^{\frac{\alpha}{k}}}\right)^{-\gamma,k}. \qquad \text{(III.3)}$$

If in (III.3) we consider $k = \gamma = 1$ we have

$$\mathcal{L}\left[e^{\lambda t}_{\alpha}\right] = \frac{1}{s^{\alpha}} \sum_{n=0}^{\infty} \left(\frac{\lambda}{s^{\alpha}}\right)^n = \frac{1}{s^{\alpha} - \lambda} \qquad \text{(III.4)}$$

(c.f. [6]).

For the convergence of the series in (III.2) see [8] and also [9].

To obtain the Laplace Transform of ${}_k\cos_{\gamma,\alpha}(\lambda t)$ and ${}_k\sin_{\gamma,\alpha}(\lambda t)$ taking into account (II.10) and (II.11) we have

$$\mathcal{L}\left[{}_k\cos_{\gamma,\alpha}(\lambda t)\right] = \frac{1}{2}\left\{\mathcal{L}\left[{}_k e^{i\lambda t}_{\gamma,\alpha}\right] + \mathcal{L}\left[{}_k e^{-i\lambda t}_{\gamma,\alpha}\right]\right\} \qquad \text{(III.5)}$$

and from (III.3) it results

$$\mathcal{L}\left[{}_k\cos_{\gamma,\alpha}(\lambda t)\right] = \frac{k^{1-\frac{\alpha}{k}}}{2s^{\alpha}} \left\{\left(1 - \frac{\lambda}{(ks)^{\frac{\alpha}{k}}}\right)^{-\gamma,k} + \left(1 + \frac{\lambda}{(ks)^{\frac{\alpha}{k}}}\right)^{-\gamma,k}\right\}. \qquad \text{(III.6)}$$

When taken $k = \alpha = \gamma = 1$; (III.6) reduces to the classical expression

$$\mathcal{L}\left[\cos(\lambda t)\right] = \frac{s}{s^2 + \lambda^2} \qquad \text{(III.7)}$$

Analogous it may be obtained

$$\mathcal{L}\left[\sin(\lambda t)\right] = \frac{s}{s^2 + \lambda^2} \qquad \text{(III.8)}$$

IV The $\mathcal{E}_j^{k,\gamma,\alpha}$ Function

Motivated by the result expressed by the formula (II.5) and looking for to be verified the classical relationship between the derivative of the sine(cosine) with the cosine(sine), by considering the variation of family of the index of the k-Pochhammer symbol, we introduce the following family of Mittag-Leffler type.

***Definition* 1** *: Let $\alpha, \gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $Re(s) > 0$, $k > 0$ and $z \in \mathbb{C} - \{0\}$, and $j \in \mathbb{N}_0$.*

$$\mathcal{E}_j^{k,\gamma,\alpha}(\lambda z) = \sum_{n\geq 0} \frac{(\gamma)_{n+j,k}\lambda^n z^{\frac{\alpha}{k}(n+1)-1}}{\Gamma_k(\alpha(n+1))(n+j)!}. \tag{IV.1}$$

It can be seen that if in (IV.1) is considered $j = 0$, is obtained the function k-α-Exponential (c.f.[7])

$${}_k e_{\gamma,\alpha}^{\lambda z} = z^{\alpha/k-1} E_{k,\alpha,\alpha}^{\gamma}\left(\lambda z^{\alpha/k}\right), \tag{IV.2}$$

When $j = 0$ and $k = 1$ is obtained the function α-Exponencial function given by (I.9).

We will demostrate that the $\mathcal{E}_j^{k,\gamma,\alpha}$ function satisfies the following properties:

1. $\mathcal{E}_0^{k,\gamma,\alpha}(\lambda z) =_k e_{\gamma,\alpha}^{\lambda z}$.
2. $D^{\alpha/k}\left[\mathcal{E}_0^{k,\gamma,\alpha}(\lambda z)\right] = \lambda k^{-\alpha/k}\mathcal{E}_1^{k,\gamma,\alpha}(\lambda z)$.
3. $D^{\alpha/k}\left[\mathcal{E}_j^{k,\gamma,\alpha}(\lambda z)\right] = \lambda k^{-\alpha/k}\mathcal{E}_{j+1}^{k,\gamma,\alpha}(\lambda z)$.
4. $\left(D^{\alpha/k}\right)^j \left({}_k e_{\gamma,\alpha}^{\lambda z}\right) = \left(\lambda k^{-\alpha/k}\right)^j \mathcal{E}_j^{k,\gamma,\alpha}(\lambda z)$.

Proof.

The first one results from (IV.1) when $j = 0$. Moreover, it suffices to prove 3) ,to see that also verified 2). Finaly we will show the property 3) by induction on the index j.

We will prove that 4) is verified. By using the definition and the relationship (I.16)

$$\begin{aligned} D^{\alpha/k}\left(\mathcal{E}_j^{k,\gamma,\alpha}(\lambda z)\right) &= \sum_{n\geq 1} \frac{(\gamma)_{n+j}\lambda^n}{\Gamma_k\left[\alpha(n+1)\right](n+j)!} D^{\alpha/k}\left(z^{\alpha/k(n+1)-1}\right) \\ &= \sum_{n\geq 1} \frac{(\gamma)_{n+j}\lambda^n}{\Gamma_k\left[\alpha(n+1)\right](n+j)!} \frac{\Gamma(\frac{\alpha}{k}(n+1))}{\Gamma(\frac{\alpha n}{k})} z^{\frac{\alpha}{k}n-1} \end{aligned}$$

Now, taking into account the relationship between the two gamma functions, the classical and k-Gamma, $\Gamma(\frac{\alpha n}{k}) = k^{1-\frac{\alpha n}{n}}\Gamma(\alpha n)$, results

$$\begin{aligned} D^{\alpha/k}\left(\mathcal{E}_j^{k,\gamma,\alpha}(\lambda z)\right) &= \sum_{n\geq 1}\frac{(\gamma)_{n+j}\lambda^n}{\Gamma_k\left[\alpha(n+1)\right](n+j)!}\frac{k^{-\frac{\alpha}{k}}\Gamma(\alpha(n+1))}{\Gamma(\alpha n)}z^{\frac{\alpha}{k}n-1} \\ &= k^{-\frac{\alpha}{k}}\sum_{n\geq 1}\frac{(\gamma)_{n+j+1}\lambda^{n+1}}{\Gamma_k\left[\alpha(n+1)\right](n+j+1)!}z^{\frac{\alpha}{k}n-1} \\ &= \lambda k^{-\alpha/k}\mathcal{E}_{j+1}^{k,\gamma,\alpha}(\lambda z). \end{aligned} \tag{IV.3}$$

We will prove the property 4) by inducction on j.

In view the properties 1) y 2) we known that this relationship is verified to $j = 1$. Suppose it holds for $j = n$, we see that this implies that equality holds for $j = n + 1$:

$$\begin{aligned} \left(D^{\alpha/k}\right)^{j+1}(\mathcal{E}_0^{k,\gamma,\alpha}) &= D^{\alpha/k}\left(D^{\alpha/k}\right)^j(\mathcal{E}_0^{k,\gamma,\alpha}) = D^{\alpha/k}\left[\left(\lambda k^{-\alpha/k}\right)^j\mathcal{E}_j^{k,\gamma,\alpha}\right] \\ &= \left(\lambda k^{-\alpha/k}\right)^j D^{\alpha/k}(\mathcal{E}_j^{k,\gamma,\alpha}) \\ &= \left(\lambda k^{-\alpha/k}\right)^{j+1}\mathcal{E}_{j+1}^{k,\gamma,\alpha}. \end{aligned} \tag{IV.4}$$

that is what we wanted to prove. □

IV.1 Trigonometric Functions

On the basis of (I.1), looking for another generalization of the classical trigonometric functions cosine and sine, we put by

Definition **2** *: Let α, $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $Re(s) > 0$, $k > 0$ and $z \in \mathbb{C} - \{0\}$ y $j \in \mathbb{N}_0$*

$$\begin{aligned} {}_j\cos_{k,\gamma,\alpha}(\lambda z) &= Re\left\{\mathcal{E}_j^{k,\gamma,\alpha}(i\lambda z)\right\} \\ {}_j\sin_{k,\gamma,\alpha}(\lambda z) &= Im\left\{\mathcal{E}_j^{k,\gamma,\alpha}(i\lambda z)\right\} \end{aligned} \tag{IV.5}$$

o equivalently

$${}_j\cos_{k,\gamma,\alpha}(\lambda z) = \sum_{n\geq 0}\frac{(-1)^n(\gamma)_{2n+j,k}\lambda^{2n}z^{\alpha/k(2n+1)-1}}{\Gamma_k\left[\alpha(2n+1)\right](2n+j)!}. \tag{IV.6}$$

$${}_j\sin_{k,\gamma,\alpha}(\lambda z) = \sum_{n\geq 0}\frac{(-1)^n(\gamma)_{2n+1+j,k}\lambda^{2n+1}z^{\alpha/k(2n+2)-1}}{\Gamma_k\left[\alpha(2n+2)\right](2n+1+j)!}. \tag{IV.7}$$

The functions introduced in (IV.5) satisfies the following properties:

Lemma: Given α, $\gamma \in \mathbb{C}$, $Re(\gamma) > 0$, $Re(s) > 0$, $k > 0$ and $z \in \mathbb{C} - \{0\}$ y $j \in \mathbb{N}_0$, then

1.
$$D^{\alpha/k}\left\{_j \sin_{k,\gamma,\alpha}(\lambda z)\right\} = \lambda k^{-\alpha/k}\ _{j+1}\cos_{k,\gamma,\alpha}(\lambda z). \tag{IV.8}$$

2.
$$D^{\alpha/k}\left\{_j \cos_{k,\gamma,\alpha}(\lambda z)\right\} = -\lambda k^{-\alpha/k}\ _{j+1}\sin_{k,\gamma,\alpha}(\lambda z). \tag{IV.9}$$

Proof. By using the definition (IV.1) and the properties demostrated in **IV.1**, we have

$$\begin{aligned} D^{\alpha/k}\left[\mathcal{E}_j^{k,\gamma,\alpha}(i\lambda z)\right] &= \lambda k^{-\alpha/k}\mathcal{E}_{j+1}^{k,\gamma,\alpha}(i\lambda z) \\ &= \lambda k^{-\alpha/k}\sum_{n\geq 0}\frac{(\gamma)_{n+j+1,k}(i\lambda)^n z^{\alpha/k(n+1)-1}}{\Gamma_k\left[\alpha(n+1)\right](n+j+1)!}. \end{aligned} \tag{IV.10}$$

Grouping of course with the powers of imaginary unit, it results

$$\begin{aligned} D^{\alpha/k}\left[\mathcal{E}_j^{k,\gamma,\alpha}(i\lambda z)\right] &= \\ &-\lambda k^{-\alpha/k}\sum_{n\geq 0}(-1)^n\frac{(\gamma)_{2n+2+j,k}\lambda^{2n+1}z^{\alpha/k(2n+2)-1}}{\Gamma_k\left[\alpha(2n+2)\right](2n+2+j)!}+ \\ &+ i\ \lambda k^{-\alpha/k}\sum_{n\geq 0}(-1)^n\frac{(\gamma)_{2n+1+j,k}\lambda^{2n}z^{\alpha/k(2n+1)-1}}{\Gamma_k\left[\alpha(2n+1)\right](2n+1+j)!}. \end{aligned} \tag{IV.11}$$

As $\mathcal{E}_j^{k,\gamma,\alpha}(i\lambda z) =_j \cos_{k,\gamma,\alpha}(\lambda z) + i_j \sin_{k,\gamma,\alpha}(\lambda z)$, results the assertion. □

References

[1] Almusharrf, A. Development of Fractional Trigonometry and an applications of Fractional Calculus to Pharmacokenetic Model. *Maters Theses and Specialist Proyects*. (2011). Paper 1048. http://digitalcommons.wku.edu/theses/1048.

[2] Bonilla, B.; Ribero, M.; Trujillo, J. Linear Differential Equations of fractional order. *J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering.* Springer. (2007).

[3] Bonilla, B.; Rivero, M. and Trujillo,J.; Fractional Differential Ecuations as alternative models to nonlinear differential equations. *Applied Mathematic and Computation*. Applied Math. and Computation. 187.(2007).

[4] Diaz, R.; Pariguan, E. On Hypergeometric Functions and k-Pochhammer symbol. *Divulgaciones Matematicas*.Vol.15 2, (2007).

[5] Dorrego, G.; Cerutti, R. The k-Mittag-Leffler Function. *Int. J. Contemp. Math. Sciences*, no 15, Vol. 7, (2012).

[6] Kilvas, H.; Srivastava, H, and Trujillo, J. *Theory and Application of Fractional Differential Equations*. Elsevier.(2006).

[7] Luque, L.; Cerutti, R.; The k-α-Exponential Function. *Int. J. of Math. Analysis. Hikari*. Vol.7, no 9-12, (2013).

[8] Prajapati J.; Shukla, A. Decomposition of Generalized Mittag-Leffler function and its properties. *Advances in Pure Math.*. No. 2. (2012).DOI: 10.4236/apm.2012.21003.

[9] Shah, P.; Jana, R. Results on Generalized Mittag-Leffler Function via Laplace Transform. *Applied Math. Sciences*. 7(12).(2013).

Received: August 1, 2014