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An The New Riemann-Liouville Fractional Operator Extended

Research Article

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Abstract: In this paper we will introduce a new and modified Riemann-Liouville fractional operator that resulted from modifying the extended fractional derivative due to M. Ozarslan ([5]). We will study some familiar functions regarding this new operator, the transform Laplace and Mellin are calculate of the potential function and we will also define a new hypergeometric function in term of extended beta function due to Pucheta ([11]).

Keywords: Extended beta function, Hypergeometric function, Fractional Calculus, Laplace and Mellin transform.

1. Introduction and Preliminaries

As it is known in 1997 Chaudhry introduced an extension of the beta function as follows: (see [8, 9])

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\,dt \tag{1}$$

Where $p \geq 0$, $R_e(x) > 0$ and $R_e(y) > 0$. As it is known in 2004 Chaudhry generalizes the hypergeometric function in term of the beta function given by: (see [8, 9])

$$F_p(a,b,c,z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!} \tag{2}$$

Where $p \geq 0$, $|z| < 1$, $R_e(c) > R_e(b) > 0$ y $(a)_n$ is the symbol Pochhammer and is defined as:

$$(a)_n = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1\ldots(a+n-1)) & \text{if } n \in \mathbb{N} \end{cases}$$

From the generalized hypergeometric function, we have the following integral representation.

$$F_p(a,b,c,z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} e^{\frac{-p}{t(1-t)}}\,dt \tag{3}$$

Where $p \geq 0$, $R_e(c) > R_e(b) > 0$ y $|arg(1-z)| < \pi$. Start recalling some definitions of elements that well be used in developing this paper.

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Definition 1.1. *Let $f \in L^1_{loc}[a,b]$, $-\infty < a \leq t \leq b < \infty$. Then, the Riemann-Liouville fractional integral of order $v > 0$ is defined as:*

$$I^v f(t) = \frac{1}{\Gamma(v)} \int_a^b (t-\lambda)^{v-1} f(\lambda) d\lambda \tag{4}$$

Definition 1.2. *Let $f \in L^1_{loc}[a,b]$, $-\infty < a \leq t \leq b < \infty$ and $m-1 \leq v < m$, $m \in \mathbb{N}$. Then, the Riemann-Liouville fractional derivative of orden v is defined as:*

$$\begin{aligned} D^v f(t) &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)} \int_a^b (t-\lambda)^{m-v-1} f(\lambda) d\lambda\right) \\ &= \frac{d^m}{dt^m}\left(I^{m-v} f(t)\right) \end{aligned} \tag{5}$$

In 2010 M. Ozarslan and E. Ozergin (see [5]) introduced an extension of the Riemann-Liouville fractional derivative and fractional integral given by the following.

Definition 1.3. *Let $f \in AC[0,b]$, be the space of functions which are absolutely continuous on $[0,b]$, $0 \leq z \leq b$, $p \geq 0$ and $v > 0$. Then, the extended Riemann-Liouville integral fractional of order $v > 0$ is defined as:*

$$I_z^{v,p} f(z) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} f(t) e^{\frac{-pz^2}{t(z-t)}} dt \tag{6}$$

Definition 1.4. *Let $f \in AC[0,b]$, be the space of functions which are absolutely continuous on $[0,b]$, $0 \leq z \leq b <$, $p \geq 0$, $v > 0$ and $m-1 \leq v < m$, $m \in \mathbb{N}$. Then, the extended Riemann-Liouville fractional derivative of order v is defined as:*

$$\begin{aligned} D_z^{v,p} f(t) &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)} \int_0^z (z-t)^{m-v-1} f(t) e^{\frac{-pz^2}{t(z-t)} dt}\right) \\ &= \frac{d^m}{dt^m}\left(I_z^{m-v,p} f(t)\right) \end{aligned} \tag{7}$$

Remark 1.5. *Note that if $p = 0$. Then, (6) and (7) reduces to the classical Riemann-Liouville fractional integral and fractional derivative of arbitrary order v (4) and (5).*

2. An The New Hypergeometric Function.

Definition 2.1. *Let $p \geq 0$, $a, b, c \in \mathbb{C}$, that such $R_e(c) > R_e(b) > 0$, $\alpha \in \mathbb{R}^+$ and $|z| < 1$. The new hypergeometric function is defined for following series:*

$$F_p^\alpha(a,b,c,z) = \sum_{n=0}^{\infty} \frac{B_p^\alpha(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!} \tag{8}$$

Where $B_p^\alpha(,)$ is the modified and extended beta function due to Pucheta (see [11]) and is defined as:

$$B_p^\alpha(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_\alpha(-bt(1-t))dt$$

Lemma 2.2. *Let $p \geq 0$, $a, b, c \in \mathbb{C}$, that such $R_e(c) > R_e(b) > 0$, $\alpha \in \mathbb{R}^+$ and $|arg(1-z)| < \pi$. Then, the new hypergeometric function it has the following integral representation:*

$$F_p^\alpha(a,b,c,z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} E_\alpha(-pt(1-t))dt \tag{9}$$

Proof. Taking into account that $(1-zt)^{-a}=\sum\limits_{n=0}^{\infty}(a)_n\frac{(tz)^n}{n!}$ and with uniform convergence, we can interchange the orden of the series:

$$\begin{aligned}
&\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}E_\alpha(-pt(1-t))dt\\
&\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{n=0}^{\infty}(a)_n\frac{(tz)^n}{n!}E_\alpha(-pt(1-t))dt\\
&\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}E_\alpha(-pt(1-t))dt\\
&\sum_{n=0}^{\infty}\frac{B_p^\alpha(b+n.c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}=F_p^\alpha(a,b,c,z)
\end{aligned}$$

□

3. An The New Extended Riemann-Liouville Fractional Operator

In this section we present the definitions and some properties of the new extended Riemann-Liouville fractional integrals and fractional derivatives of the potential function.

Definition 3.1. *Let $f\in AC[0,b]$, $0\le z\le b$, $p\ge 0$, $\alpha\in\mathbb{R}^+$ and $v>0$. Then, the new extended Riemann-Liouville fractional integral of order $v>0$ is defined as:*

$$I_z^{v,p,\alpha}f(z)=\frac{1}{\Gamma(v)}\int_0^z(z-t)^{v-1}f(t)E_\alpha\left(\frac{-pt(z-t)}{z^2}\right)dt \tag{10}$$

Definition 3.2. *Let $f\in AC[0,b]$, $0\le z\le b<$, $p\ge 0$, $\alpha\in\mathbb{R}^+$ $v>0$ and $m-1\le v<m$, $m\in\mathbb{N}$. Then, the new extended Riemann-Liouville derivative fractional of order v is defined as:*

$$\begin{aligned}
D_z^{v,p,\alpha}f(t)&=\frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)}\int_0^z(z-t)^{m-v-1}f(t)E_\alpha\left(\frac{-pt(z-t)}{z^2}\right)\right)dt\\
&=\frac{d^m}{dt^m}\left(I_z^{m-v,p}f(t)\right) \qquad (11)
\end{aligned}$$

Remark 3.3. *Note that if $\alpha=1$. Then, (10) and (11) is reduced to extended Riemann-Liouville fractional derivative and fractional integral (6) and (7).*

Theorem 3.4. *Let $p\ge 0$, $\alpha\in\mathbb{R}^+$, $v>0$ and $f(z)=z^\lambda$, $\lambda>0$. Then*

$$I_z^{v,p,\alpha}f(z)=\frac{B_p^\alpha(\lambda+1,v)}{\Gamma(v)}z^{\lambda+v} \tag{12}$$

Proof.

$$I_z^{v,p,\alpha}\left(z^\lambda\right)=\frac{1}{\Gamma(v)}\int_0^z(z-t)^{v-1}z^\lambda E_\alpha\left(\frac{-pt(z-t)}{z^2}\right)dt \tag{13}$$

Making a variable change $u=\frac{t}{z}$, we have: $dt=zdu$; $t=0,\ u=0$; $t=z,\ u=1$; $(z-t)=z(1-u)$. Thus, replacing in the previous expression (13), we have:

$$\begin{aligned}
I_z^{v,p,\alpha}\left(z^\lambda\right)&=\frac{1}{\Gamma(v)}z^{\lambda+v}\int_0^1 u^\lambda(1-u)^{v-1}E_\alpha\left(-pu(1-u)\right)du\\
&=\frac{B_p^\alpha(\lambda+1,v)}{\Gamma(v)}z^{\lambda+v}
\end{aligned}$$

□

Theorem 3.5. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^+$ $v > 0$ *and* $m-1 \leq v < m$, $m \in \mathbb{N}$ *and* $f(z) = z^{\lambda}$ $\lambda > 0$. *Then*

$$D_z^{v,p,\alpha}(z^{\lambda}) = \frac{B_p^{\alpha}(\lambda+1, m-v)}{\Gamma(m-v)} \frac{\Gamma(\lambda+m-v+1)}{\Gamma(\lambda-v+1)} z^{\lambda-v} \tag{14}$$

Proof. From Definition (11) and (12), we have:

$$\begin{aligned} D_z^{v,p,\alpha} z^{\lambda} &= \frac{d^m}{dt^m}\left(I_z^{m-v,p} z^{\lambda}\right) = \frac{d^m}{dt^m}\left(\frac{B_p^{\alpha}(\lambda+1,m-v)}{\Gamma(m-v)} z^{\lambda+m-v}\right) \\ &= \frac{B_p^{\alpha}(\lambda+1,m-v)}{\Gamma(m-v)} \frac{d^m}{dz^m} z^{\lambda+m-v} \\ &= \frac{B_p^{\alpha}(\lambda+1,m-v)}{\Gamma(m-v)} \frac{\Gamma(\lambda+m-v+1)}{\Gamma(\lambda-v+1)} z^{\lambda-v} \end{aligned}$$

$\square$

Theorem 3.6. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$ *such that* $v - \lambda > 0$, $|z| < 1$ *and* $f(z) = z^{\lambda-1}(1-z)^{\beta-1}$, $\lambda > 0$ $\beta > 0$. *Then*

$$I_z^{v-\lambda,p,\alpha} f(z) = \frac{\Gamma(\lambda) F_p^{\alpha}(\beta,\lambda,v,z)}{\Gamma(v)} z^{v-1} \tag{15}$$

Proof. From Definitions (10), (9) and using $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we have

$$I_z^{v-\lambda,p,\alpha}\left(z^{\lambda-1}(1-z)^{\beta}\right) = \frac{1}{\Gamma(v-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{\beta-1}(z-t)^{v-\lambda-1} \times E_{\alpha}\left(\frac{-pt(z-t)}{z^2}\right) dt \tag{16}$$

Marking the change of variable $u = \frac{t}{z}$, we have: $dt = zdu$; $t = 0$, $u = 0$; $t = z$, $u = 1$. Thus, replacing in the previous expression (16), we have:

$$\begin{aligned} I_z^{v-\lambda,p,\alpha}\left(z^{\lambda-1}(1-z)^{\beta}\right) &= \frac{1}{\Gamma(v-\lambda)} \int_0^1 t^{\lambda-1} z^{\lambda-1}(1-uz)^{\beta} z^{v-\lambda-1}(1-u)^{v-\lambda-1} \times E_{\alpha}\left(-pu(1-u)\right) zdu \\ &= \frac{z^{v-1}}{\Gamma(v-\lambda)} \int_0^1 u^{\lambda-1}(1-u)^{v-\lambda-1}(1-uz)^{\beta} \times E_{\alpha}\left(-pu(1-u)\right) zdu \\ &= \frac{z^{v-1}}{\Gamma(v-\lambda)} B(\lambda, v-\lambda) F_p^{\alpha}(\beta,\lambda,v,z) \\ &= \frac{z^{v-1}\Gamma(\lambda)}{\Gamma(v)} F_p^{\alpha}(\beta,\lambda,v,z) \end{aligned}$$

$\square$

Theorem 3.7. *Let* $f(z)$ *be an analytic function in the disc* $|z| < \rho$, $\rho > 0$, *and has the power series expansion* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. *Then*

$$I_z^{v,p,\alpha}\left(z^{\lambda-1} f(z)\right) = \frac{z^{\lambda+v-1}}{\Gamma(v)} \sum_{n=0}^{\infty} a_n B_p^{\alpha}(\lambda+n, v) z^n \tag{17}$$

Proof. Using the Definition (10) and taking variable change $u = \frac{t}{z}$, we obtain:

$$\begin{aligned} I_z^{v,p,\alpha}\left(z^{\lambda-1} f(z)\right) &= \frac{1}{\Gamma(v)} \int_0^z t^{\lambda-1} \sum_{n=0}^{\infty} a_n t^n (z-t)^{v-1} E_{\alpha}\left(\frac{-pt(z-t)}{z^2}\right) dt \\ &= \sum_{n=0}^{\infty} \frac{z^{\lambda+v+n-1}}{\Gamma(v)} \int_0^1 u^{\lambda+n-1}(1-u)^{v-1} E_{\alpha}(-pu(1-u)) du \\ &= \sum_{n=0}^{\infty} a_n \frac{z^{\lambda+v+n-1}}{\Gamma(v)} B_p^{\alpha}(\lambda+n, v) \\ &= \frac{z^{\lambda+v-1}}{\Gamma(v)} \sum_{n=0}^{\infty} a_n B_p^{\alpha}(\lambda+n, v) \end{aligned}$$

$\square$

4. Transform Integral

In this section we will evaluate the Laplace and Mellin transform of the new extended Riemann-Liouville fractional integrals and fractional derivatives of the potential function $f(z) = z^w \quad w > 0$

Definition 4.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ an exponential order function and piecewise continuous, then the Laplace transform of f is*

$$\mathcal{L}\left\{f(t)\right\}(s) = \int_0^\infty e^{-st} f(t)dt \quad s \in \mathbb{C} \tag{18}$$

The integral exist for $R_e(s) > 0$.

Definition 4.2. *The Mellin transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is defined by*

$$M\left\{f(t)\right\}(s) = \int_0^\infty t^{s-1} f(t)dt \quad s \in \mathbb{C} \tag{19}$$

4.1. Transform Laplace

Theorem 4.3. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$ and $f(z) = z^w$, $w > 0$. Then*

$$\mathcal{L}\left\{I_z^{v,p,\alpha} z^w\right\}(s) = \frac{B_p^\alpha(w+1,v)}{\Gamma(v)} \frac{\Gamma(w+2)}{s^{w+2}} \tag{20}$$

Proof. From (18), (10) and making variable change $u = \frac{t}{z}$, we have

$$\begin{aligned}
\mathcal{L}\left\{I_z^{v,o,\alpha} z^w\right\}(s) &= \int_0^\infty e^{-st}\left(\frac{1}{\Gamma(v)}\int_0^z t^w (z-t)^{v-1} E_\alpha\left(\frac{-pt(z-t)}{z^2}\right) d\lambda\right) dt \\
&= \frac{1}{\Gamma(v)}\int_0^\infty e^{-st} t^{w+1}\left(\int_0^1 u^w (1-u)^{v-1} E_\alpha(-pu(1-u))du\right) dt \\
&= \frac{B_p^\alpha(w+1,v)}{\Gamma(v)}\int_0^\infty e^{-st} t^{w+1} dt \\
&= \frac{B_p^\alpha(w+1,v)}{\Gamma(v)} \frac{\Gamma(w+2)}{s^{w+2}}
\end{aligned}$$

□

Theorem 4.4. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$ $v > 0$ and $m-1 \leq v < m$, $m \in \mathbb{N}$ and $f(z) = z^w \quad w > 0$. Then*

$$\mathcal{L}\left\{D_z^{v,p,\alpha} z^w\right\}(s) = \frac{B_p^\alpha(w+1,m-v)}{\Gamma(m-v)} \frac{\Gamma(w+2)}{s^{w-m+2}} \tag{21}$$

Proof. Using Definition (11) and (20), it result

$$\begin{aligned}
\mathcal{L}\left\{D_z^{v,p,\alpha} z^w\right\}(s) = \mathcal{L}\left\{D^m\left(I_z^{m-v,p,\alpha} z^w\right)\right\}(s) = s^m \mathcal{L}\left\{I_z^{m-v,p,\alpha} z^w\right\}(s) - \underbrace{\sum_{i=1}^m s^{m-i} I_z^{i-v,p,\alpha} 0^w}_{=0} \\
= \frac{B_p^\alpha(w+1,m-v)}{\Gamma(m-v)} \frac{\Gamma(w+2)}{s^{w-m+2}}
\end{aligned}$$

□

Remark 4.5. *Note that if $p = 0$, $m = 2$, $\alpha = v = 1$ the expression (20), (21) is reduced to the classic Laplace transform of the integral and derivative of order 2 of the potential function $z^w \quad w > 0$.*

4.2. Mellin Transform

Theorem 4.6. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$ *and* $f(z) = z^w$, $w > 0$. *Then*

$$M\left\{I_z^{v,p,\alpha} z^w\right\}(s) = \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^{w+v} B(w-s+3, v-s) \tag{22}$$

Proof.

$$\begin{aligned} M\left\{I_z^{v,p,\alpha} z^w\right\}(s) &= \int_0^\infty p^{s-1}\left(\frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1} t^\lambda E_\alpha\left(\frac{-p}{z^2} t(z-t)\right) dt\right) dp \\ &= \frac{z^{v-1}}{\Gamma(v)}\int_0^\infty p^{s-1}\int_0^z \left(1-\frac{t}{z}\right)^{v-1} t^\lambda E_\alpha\left(\frac{-p}{z^2} t(z-t)\right) dtdp \end{aligned}$$

Taking variable change $u = \frac{t}{z}$, we have: $dt = zdu$; $t = 0$, $u = 0$; $t = z$, $u = 1$

$$\begin{aligned} M\left\{I_z^{v,p,\alpha} z^w\right\}(s) &= \frac{z^{v+\lambda}}{\Gamma(v)}\int_0^\infty p^{s-1}\int_0^1 u^\lambda (1-u)^{v-1} E_\alpha - pu(1-u) dudp \\ &= \frac{z^{v+\lambda}}{\Gamma(v)}\int_0^1 u^\lambda (1-u)^{v-1}\int_0^\infty p^{s-1} E_\alpha - pu(1-u) dpdu \end{aligned}$$

Thus, if call $r = pu(1-u)$, we have: $dr = u(1-u)dp$; $p^{s-1} = \frac{r^{s-1}}{u^{s-1}(1-u)^{s-1}}$

$$\begin{aligned} M\left\{I_z^{v,p,\alpha} z^w\right\}(s) &= \frac{z^{v+\lambda}}{\Gamma(v)}\int_0^1 u^{(\lambda-s+1)+1}(1-u)^{v-s+1}\int_0^\infty r^{s-1} E_\alpha(-r) drdu \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^{v+\lambda} B(\lambda-s+3, v-s) \end{aligned}$$

☐

Theorem 4.7. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$, $a \in \mathbb{C}$ *such that* $R_e(a) > 0$ *and* $|z| < 1$. *Then*

$$M\left\{I_z^{v,p,\alpha}(1-z)^{-a}\right\}(s) = \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(-s+3, v-s) F(a, -s+3, v-2s+3, z) \tag{23}$$

Proof. Taking into account that $(1-z)^{-a} = \sum\limits_{n=0}^\infty (a)_n \frac{z^n}{n!}$, the Mellin transform is a linear operator and (22), we have

$$\begin{aligned} M\left\{I_z^{v,p,\alpha}(1-z)^{-a}\right\}(s) &= \sum_{n=0}^\infty \frac{(a)_n}{n!} M\left\{I_z^{v,p,\alpha} z^n\right\}(s) \\ &= \sum_{n=0}^\infty \frac{(a)_n}{n!}\frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(n-s+3, v-s) z^n \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v \sum_{n=0}^\infty (a)_n B(n-s+3, v-s)\frac{z^n}{n!} \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(n-s+3, v-s) \times F(a, -s+3, v-2s+3, z) \end{aligned}$$

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