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On the p-k-Mittag-Leffler Function

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Abstract

In this paper, we define the function ${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z)$, estudy its analytic properties, some elementary properties as its integral expression, its relationship with the fractional operator of Riemann-Liouville and investigate the fractional generalization of the kinetic equation involving this Mittag-Leffler type function. This new Mittag-Leffler-type function related to the two parameters Gamma function recently introduced by Gehlot [8], is a generalizaton of the k-Mittag-Leffler function since ${}_k\Gamma_k(z) = \Gamma_k(z)$, thus generalize the Mittag-Leffler of two parameter, Mittag-Leffler function and the exponential function.

Keywords: p-k-Mittag-Leffler function

I Introduction

The branch of mathematical analysis known as the Fractional Calculus deals essentially with operators of differentiation and integration of non-integer orders (see e.g. [22], [14]). Many authors have generalized differential equations (integral equations) by replacing the ordinary derivative (integral) by some of the definitions that contemplate non-integers; for example: Riemann-Liouville, Caputo, Grundwald-Letnikov, Hadamard and other more modern ones such as Caputo-Fabrizio (see e.g. [22], [2], [12], [24], [20], [1], and the references in them). In the solution of such differential (integral) equations the Mittag-Leffler function naturally appears to play a role analogous to that of the exponential function in the ordinary case.

The Mittag-Leffler function has generated much interest among researchers and has been the subject of many generalizations (see e.g. [21], [15], [5], [7], [4], [26], and the references in them) giving rise to the so-called Mittag-Leffler type functions. It has also been used to generalize fractional integral operators (see e.g. [27], [6]).

II Preliminaries.

In this section we present results and definitions known and important for the development of the following sections. We begin with the Mittag-Leffler function and some of its generalizations.

The Mittag-Leffler function $E_\alpha(z)$ and its first generalization to two parameters $E_{\alpha,\beta}(z)$ are given by the following series:

$$E_\alpha(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z \in \mathbb{C};\ \mathfrak{Re}(\alpha) > 0) \tag{II.1}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z \in \mathbb{C};\ \mathfrak{Re}(\alpha) > 0, \mathfrak{Re}(\beta) > 0) \tag{II.2}$$

respectively; where $\Gamma(z)$ is the classical Gamma function, and verifies $E_{\alpha,1}(z) = E_\alpha(z)$.

Prabhakar (see [21]) introduced the Mittag-Leffler type function $E^\gamma_{\alpha,\beta}(z)$ defined by

$$E^\gamma_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} \tag{II.3}$$

with α, β and $\gamma \in \mathbb{C}$; and $(\gamma)_n$ denotes the Pochhammer symbol, and verifies $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

A generalization of the Mittag-Leffler function called k-Mittag-Leffler function has been introduced and studied in [4]:

$$E^\gamma_{k,\alpha,\beta}(z) = \sum_{n\geq 0} \frac{(\gamma)_{n,k} z^n}{\Gamma_k(\alpha n + \beta) n!} \tag{II.4}$$

with $k > 0$; α, β,$\gamma \in \mathbb{C}$; $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, and $z \in \mathbb{C}$; where Γ_k is the k-Gamma Funcion and $(\gamma)_{n,k}$ is the Pochhammer k-symbol due to Diaz and Pariguan (see [3]), and are

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k)...(\gamma + (n-1)k); \tag{II.5}$$

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \quad (\mathfrak{Re}(z) > 0). \tag{II.6}$$

The following properties are verified:

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)} \tag{II.7}$$

and

$$\Gamma_k(z) = k^{1-\frac{z}{k}} \Gamma\left(\frac{z}{k}\right), \tag{II.8}$$

with $z \in \mathbb{C} \setminus k\mathbb{Z}^-$, $k > 0$. Also, the expression $E^\lambda_{1,\alpha,\beta}(z) = E^\lambda_{\alpha,\beta}(z)$ is verified (see [4]).

Recently, K.S. Gehlot [8] has introduced a modification of the k-Gamma function by means of the following integral

$$_p\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{p}} t^{z-1} dz, \quad \text{for}\, z \in \mathbb{C} \backslash k\mathbb{Z}^-;\ k, p \in \mathbb{R}^+ \setminus \{0\},\ \mathfrak{Re}(z) > 0. \tag{II.9}$$

Also, he has defined a new Pochhammer symbol

$$\begin{aligned} _p(z)_{n,k} &= \left(\frac{zp}{k}\right) \cdot \left(\frac{zp}{p} + p\right) \cdot \left(\frac{zp}{k} + 2p\right) ... \left(\frac{zp}{k} + (n-1)p\right) \\ &= \frac{_p\Gamma_k(z+nk)}{_p\Gamma_k(z)} \end{aligned} \tag{II.10}$$

and also establishes the relations between the function introduced by him and the k-Gamma and the classical Gamma function. In fact, we have following.

Lemma 1. *For the p-k-Gamma function, the k-Gamma function and the classical Gamma function it is verified:*

$$_p\Gamma_k(z) = \left(\frac{p}{k}\right)^{\frac{z}{k}} \Gamma_k(z) = \frac{p^{\frac{z}{k}}}{k} \Gamma\left(\frac{z}{k}\right) \tag{II.11}$$

The proof could be seen in [8].

And, for the $_p(z)_{n,k}$ Pochhammer symbol, we have the following relation

Lemma 2. *For the p-k-Pochhammer symbol, the k-Pochhammer symbol and the classical Pochhammer symbol it has*

$$_p(z)_{n,k} = \left(\frac{p}{k}\right)^n (z)_{n,k} = p^n \left(\frac{z}{k}\right)_n \tag{II.12}$$

Definition **1.** *Let $[a,b] \subset \mathbb{R}$ be $(-\infty < a < b < \infty)$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{R}$, with $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, are defined by*

$$(I^{\alpha}_{+a}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (x > a \ ; \quad \alpha > 0) \tag{II.13}$$

and

$$(I^{\alpha}_{-b}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt \quad (x < b \ ; \quad \alpha > 0), \tag{II.14}$$

respectively.

The Riemann-Liouville fractional derivative $D^{\alpha}_{a+}f$ and $D^{\alpha}_{b-}f$ of order $\alpha > 0$ are defined by

$$(D^{\alpha}_{a+}f)(x) = \left(\frac{d}{dx}\right)^m (I^{m-\alpha}_{a+}f)(x) \quad (x > a) \tag{II.15}$$

and

$$(D^{\alpha}_{b-}f)(x) = \left(-\frac{d}{dx}\right)^m (I^{m-\alpha}_{-b}f)(x) \quad (x < b), \tag{II.16}$$

respetively.

In particular, when $\alpha = m \in \mathbb{N}_0$, then

$$(D^{0}_{a+}f)(x) = (D^{0}_{b-})(x) = f(x), \tag{II.17}$$

$$(D^{m}_{a+}f)(x) = f^{(m)}(x), \tag{II.18}$$

$$(D^{m}_{b-})(x) = (-1)^n f^{(m)}(x) \tag{II.19}$$

(see, for example, [14]).

In 2012 Mubeen and Habbibulah (see [19]) introduced the k-Riemann-Liouville fractional integral given by the following

Definition **2.** *Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L^1([0,\infty))$. Then the k-Riemann-Liouville fractional integral of f is*

$$I^{\alpha}_{k}f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau, \ (t > 0); \tag{II.20}$$

wich can be written as a convolution with the singular kernel of Riemann-Liouville:

$$I^{\alpha}_{k}f(t) = \frac{t^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} * f(t), \ t > 0. \tag{II.21}$$

Definition **3** (**Laplace transform**)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}$ an exponential order and piecewise continuous function, then the Laplace transform of f is*

$$\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t) dt. \tag{II.22}$$

The integral exist for $\mathfrak{Re}(s) > 0$.

III Definition and convergence conditions

In view of the expressions (II.9) and (II.10) we introduce a new function of Mittag-Leffler type that we will call the p-k-Mittag-Leffler function by means of the following

<u>Definition</u> **4.** *Let α, β, γ be complex numbers with $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, $\mathfrak{Re}(\gamma) > 0$; and let $p, k \in \mathbb{R}^+ \setminus \{0\}$. We define the p-k-Mittag-Leffler function ${}_p\mathbb{E}_{k,\alpha,\beta}(z)$ by the series*

$$ {}_p\mathbb{E}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{k,n}\, z^n}{{}_p\Gamma_k(\alpha n + \beta) n!} \tag{III.1}$$

where ${}_p\Gamma_k(x)$ is given by (II.9), and ${}_p(\gamma)_{n,k}$ is the Pochhammer symbol given by (II.10).

As particular cases of ${}_p\mathbb{E}_{k,\alpha,\beta}(z)$ we have the classical two parameters Mittag-Leffler function $E_{\alpha,\beta}(z)$ for $p = k = \gamma = 1$ and for $p = k$ one gets the k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$, and for $p = k = 1$, the Mittag-Leffler function due to Prabhakar [21] is obtained.

III.1 Order and Type of the ${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z)$ function

In this section, we present some caracteristics of the p-k-Mittag-Leffler function. We will show that the p-k-Mittag-Leffler function is an entire function, and we will also give the order and the type.

Theorem 1. *The p-k-Mittag-Leffler function, defined in (III.1), is an entire function of order ρ and type σ given by*

$$\rho = \frac{k}{\mathfrak{Re}(\alpha)} \ , \quad and \quad \sigma = \left[\rho p e^{\mathfrak{Re}(\alpha/k \ln(\alpha/k))\rho}\right]^{-1}. \tag{III.2}$$

Moreover, for all $\epsilon > 0$,the next asymptotic estimate holds:

$$\left| {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z)\right| < e^{(\sigma+\epsilon)|z|^{\rho}}, \tag{III.3}$$

with ρ, σ as in (III.2), for $|z| \geq r_0$, $r_0 = r_0(\epsilon) > 0$ sufficiently large.

Proof. The radius of convergence of the p-k-Mittag-Leffler function will be called R. Taking into account the properties (II.10) and (II.11),and making use of the asymptotic expansions for the Gamma function(see for example [14]); and the asymptotic Stirling's formula:

$$\Gamma(z) = (2\pi)^{\frac{1}{2}} z^{z-\frac{1}{2}} e^{-z}[1 + O(z-1)] \quad (|\arg(z)| < \pi \ ; \ |z| \to \infty); \tag{III.4}$$

in particular,

$$n! = (2\pi n)^{\frac{1}{2}} n^n e^{-n}[1 + O(n^{-1})] \quad (n \in \mathbb{N}\ ;\ n \to \infty); \tag{III.5}$$

and the quotient expansion of two Gamma functions at infinity:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}[1 + O(z^{-1})] \quad (|\arg(z) + a| < \pi\ ;\ |z| \to \infty). \tag{III.6}$$

Rewriting the series (III.1) in the following way

$${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n + \beta) n!} z^n = \sum_{n=0}^{\infty} c_n z^n. \tag{III.7}$$

Since

$$R = \limsup_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| ; \tag{III.8}$$

in view of the properties (III.4), (III.5) and (III.6), we can easily see that

$$\begin{aligned}
\left|\frac{c_n}{c_{n+1}}\right| &= \left| \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta) n!} \frac{{}_p\Gamma[\alpha(n+1)+\beta](n+1)!}{{}_p(\gamma)_{n+1,k}} \right| \\
&= \left| \frac{{}_p\Gamma_k(\gamma+nk)}{{}_p\Gamma_k(\gamma)\ {}_p\Gamma_k(\alpha n+\beta) n!} \frac{{}_p\Gamma_k(\gamma)\ {}_p\Gamma_k(\alpha n+\alpha+b)(n+1)!}{{}_p\Gamma_k[\gamma+(n+1)k]} \right| \\
&= \left| \frac{k^{-1} p^{\frac{\gamma+nk}{k}} \Gamma\left(\frac{\gamma+nk}{k}\right)}{k^{-1} p^{\frac{\alpha n+\beta}{k}} \Gamma\left(\frac{\alpha n++\beta}{k}\right) n!} \frac{k^{-1} p^{\frac{\alpha n+\alpha+\beta}{k}} \Gamma\left(\frac{\alpha n+\alpha+\beta}{k}\right)(n+1)!}{k^{-1} p^{\frac{\alpha+nk+k}{k}} \Gamma\left(\frac{\alpha+nk+k}{k}\right)} \right| \\
&= (n+1)\left|p^{\frac{\alpha}{k}-1}\right| \left| \frac{\Gamma\left(\frac{\alpha}{k}n+\frac{\alpha}{k}+\frac{b}{k}\right)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)} \right| \left| \frac{\Gamma\left(n+\frac{\gamma}{k}\right)}{\Gamma\left(n+\frac{\gamma}{k}+1\right)} \right| \\
&\approx \left|p^{\frac{\alpha}{k}-1}\right| \left| \left(\frac{\alpha}{k}n\right)^{\frac{\alpha}{k}} \right| \to \infty.
\end{aligned} \tag{III.9}$$

Thus, the p-k-Mittag-Leffler function is an entire function.
To determine the order ρ and the type σ are applied the next definitions:

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(1/|c_n|)}, \tag{III.10}$$

$$e\rho\sigma = \limsup_{n\to\infty} \left(n|c_n|^{\frac{\rho}{n}} \right). \tag{III.11}$$

Using the relations (III.4), (III.5), (III.6) and (III.7) we have

$$\begin{aligned}
\frac{1}{|c_n|} &= \left|\frac{{}_p\Gamma_k(\gamma)\ {}_p\Gamma_k(\alpha n+\beta)n!}{{}_p\Gamma_k(\gamma+nk)}\right| \\
&= k^{-1}n!\left|p^{n\left(\frac{\alpha}{k}-1\right)+\frac{\beta}{k}}\right|\left|\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)\right|\left|\Gamma\left(\frac{\gamma}{k}+n\right)\right|^{-1}\left|\Gamma\left(\frac{\gamma}{k}\right)\right| \\
&\approx k^{-1}\left((2\pi)^{\frac{1}{2}}n^{n+\frac{1}{2}}e^{-n}\right)\left|p^{n\left(\frac{\alpha}{k}-1\right)+\frac{\beta}{k}}\right|\left(\sqrt{2\pi}\left|\left(\frac{\alpha}{k}n\right)^{\frac{\alpha}{k}n+\frac{\beta}{k}-\frac{1}{2}}\right|\left|e^{-\frac{\alpha}{k}n}\right|\right)\times \\
&\quad \times\left(\sqrt{2\pi}\left|n^{\frac{\gamma}{k}+n-\frac{1}{2}}\right|\left|\Gamma\left(\frac{\gamma}{k}\right)\right|\right)^{-1}\left|\Gamma\left(\frac{\gamma}{k}\right)\right| \\
&= k^{-1}(2\pi)^{\frac{1}{2}}\left|p^{n\left(\frac{\alpha}{k}-1\right)+\frac{\beta}{k}}\right|\left|\left(\frac{\alpha}{k}\right)^{\frac{\alpha}{k}n+\frac{\beta}{k}-\frac{1}{2}}\right|\left|n^{n\frac{\alpha}{k}+\frac{\beta}{k}-\frac{\gamma}{k}+\frac{1}{2}}\right|\left|\Gamma\left(\frac{\alpha}{k}\right)\right|e^{-n\mathfrak{Re}\left(\frac{\alpha}{k}\right)} \\
&\approx k^{-1}(2\pi)^{\frac{1}{2}}\left|p^{n\left(\frac{\alpha}{k}-1\right)}\right|\left|\left(\frac{\alpha}{k}\right)^{\frac{\alpha}{k}n}\right|\left|n^{n\frac{\alpha}{k}}\right|\left|\Gamma\left(\frac{\alpha}{k}\right)\right|e^{-n\mathfrak{Re}\left(\frac{\alpha}{k}\right)} \\
&= e^{\frac{1}{2}\ln\left(\frac{2\pi}{k}\right)+\ln\left|\Gamma\left(\frac{\alpha}{k}\right)\right|+\mathfrak{Re}\left[n\left(\frac{\alpha}{k}-1\right)\ln p+n\frac{\alpha}{k}\ln\frac{\alpha}{k}+n\frac{\alpha}{k}\ln n-n\frac{\alpha}{k}\right]}. \qquad \text{(III.12)}
\end{aligned}$$

Hence, using (III.12), by application of (III.10), we have

$$\frac{1}{\rho}=\limsup_{n\to\infty}\frac{\frac{1}{2}\ln\left(\frac{2\pi}{k}\right)+\ln\left|\Gamma\left(\frac{\alpha}{k}\right)\right|+\mathfrak{Re}\left[n\left(\frac{\alpha}{k}-1\right)\ln p+n\frac{\alpha}{k}\ln\frac{\alpha}{k}+n\frac{\alpha}{k}\ln n-n\frac{\alpha}{k}\right]}{n\ln n}, \qquad \text{(III.13)}$$

$$=\frac{\mathfrak{Re}(\alpha)}{k}$$

which is the wanted result in (III.2).

Similarly, on substituting (III.12) into (III.11), this yields

$$\begin{aligned}
\sigma e\rho &= \limsup_{n\to\infty}\left[n|c_n|^{\rho/n}\right] \\
&= \limsup_{n\to\infty}\left[e^{\ln n-\frac{1}{2}\ln\left(\frac{2\pi}{k}\right)-\ln\left|\Gamma\left(\frac{\alpha}{k}\right)\right|-\mathfrak{Re}\left[n\left(\frac{\alpha}{k}-1\right)\ln p+n\frac{\alpha}{k}\ln\frac{\alpha}{k}+n\frac{\alpha}{k}\ln n-n\frac{\alpha}{k}\right]\frac{k}{n\mathfrak{Re}(\alpha)}}\right] \\
&= p^{-1}e^{-\mathfrak{Re}\left[\frac{\alpha}{k}\ln\frac{\alpha}{k}\right]\frac{k}{\mathfrak{Re}(\alpha)}}. \qquad \text{(III.14)}
\end{aligned}$$

Finally, we obtain that

$$\sigma=\left[\rho p e^{\mathfrak{Re}(\alpha/k\ln(\alpha/k))\rho}\right]^{-1}. \qquad \text{(III.15)}$$

□

III.2 Elementary properties.

According to the Defintion 4, the following relationships can be obtained, which we will gather in the following

Theorem 2. *Let ${}_p\mathbb{E}_{k,\alpha,\beta}(z)$ be the p-k-Mittag-Leffler function given by (III.1), we have*

1.

$$ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z) = \frac{1}{{}_p\Gamma_k(\beta)} + z\sum_{n=0}^{\infty} \frac{(\gamma)_{n+1}\, z^n}{{}_p\Gamma_k(\alpha n+\alpha+\beta)(n+1)!}. \tag{III.16} $$

2.

$$ \frac{1}{p}\,{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z) = \frac{\beta}{k}\,{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+k}(z) + \frac{\alpha}{k} z\frac{d}{dz}\left({}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+k}(z)\right). \tag{III.17} $$

3.

$$ \frac{d^m}{dz^m}[z^{\frac{\beta}{k}-1}\,{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z^{\frac{\alpha}{k}})] = p^{-m} z^{\frac{\beta}{k}-m-1}\,{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta-mk}(z^{\frac{\alpha}{k}}). \tag{III.18} $$

Proof. From the (III.1), one has 1).

As particular case, if $p=k=\gamma=1$, (III.16) reduces to (5.1) f.1 from [13]:

$$ E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z) \tag{III.19} $$

To obtain (III.17), and taking into account that

$$ \begin{aligned} {}_p\Gamma_k(\alpha n\beta+k) &= \frac{p^{\frac{\alpha n+\beta+k}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta}{k}+1\right) = \frac{p^{\frac{\alpha n+\beta+k}{k}}}{k}\left(\frac{\alpha n+\beta}{k}\right)\Gamma\left(\frac{\alpha n+\beta}{k}\right) \\ &= \frac{p^{\frac{\alpha n+\beta}{k}+1}}{k}\left(\frac{\alpha n+\beta}{k}\right){}_p\Gamma_k(\alpha n+\beta) = p\,{}_p\Gamma_k(\alpha n+\beta). \end{aligned} \tag{III.20} $$

Then

$$ \frac{\alpha}{k} z\frac{d}{dz}\left({}_p\mathbb{E}_{k,\alpha,\beta+k}(z)\right) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{k,n}\frac{n\alpha}{k} z^n}{{}_p\Gamma_k(\alpha n+\beta+k)} \tag{III.21} $$

and

$$ \frac{\beta}{k}\,{}_p\mathbb{E}_{k,\alpha,\beta+k}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{k,n}\,\frac{\beta}{k} z^n}{{}_p\Gamma_k(\alpha n+\beta+k)}. \tag{III.22} $$

From (III.21) and (III.22), we have

$$ \frac{\alpha}{k} z\frac{d}{dz}\left({}_p\mathbb{E}_{k,\alpha,\beta+k}(z)\right) + \frac{\beta}{k}\,{}_p\mathbb{E}_{k,\alpha,\beta+k}(z) = \frac{1}{p}\,{}_p\mathbb{E}_{k,\alpha,\beta}(z). \tag{III.23} $$

Particular case:

If $p=k=\gamma=1$, (III.23) is

$$ \alpha z\frac{d}{dz}\left(E_{\alpha,\beta+1}(z)\right) + \beta E_{\alpha,\beta+1}(z) = E_{\alpha,\beta}(z), \tag{III.24} $$

which coincides with formulae (5.1) f.2 from [13].

To prove (III.18), we will start by showing that

$$\left(\frac{d}{dz}\right)^m \left(z^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}\right) = \left(\frac{\alpha}{k}n + \frac{\beta}{k} - 1 - (m-1)\right)_m z^{\frac{\alpha}{k}n+\frac{\beta}{k}-1-m}. \tag{III.25}$$

Knowing that

$$\left(\frac{\alpha}{k}n + \frac{\beta}{k} - 1 - (m-1)\right)_m = \frac{\Gamma\left(\frac{\alpha}{k}n + \frac{\beta}{k}\right)}{\Gamma\left(\frac{\alpha}{k}n + \frac{\beta}{k} - m\right)} \tag{III.26}$$

and, by using (III.24), it results

$$\left(\frac{\alpha}{k}n + \frac{\beta}{k} - 1 - (m-1)\right)_m = p^{-m}\frac{{}_p\Gamma_k(\alpha n + \beta)}{{}_p\Gamma_k(\alpha n + \beta - mk)}. \tag{III.27}$$

Then, from (III.25), (III.26) and (III.27) we have (III.18).

Particular cases:

- When $p = k$, (III.18) coincides with (II.28) from [4].
- When $p = k = 1$, (III.18) coincides with (11.5) from [13].
- When $p = k = \gamma = 1$, (III.18) coincides with (5.1) f.3 from [13].

□

Theorem 3. *Let $\alpha, \beta,$ and γ be complex numbers, $k > 0$, $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$ and $\mathfrak{Re}(\gamma) > 0$. Then*

$$\int_0^z t^{\frac{\beta}{k}-1}{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(at^{\frac{\alpha}{k}})dt = pz^{\frac{\beta}{k}}{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+k}(az^{\frac{\alpha}{k}}) \tag{III.28}$$

Proof. By interchanging the order of integration and summation, we have

$$\int_0^z t^{\frac{\beta}{k}-1}{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(at^{\frac{\alpha}{k}})dt = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{k,n}a^n}{{}_p\Gamma_k(\alpha n + \beta)n!} \int_0^z t^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}dt. \tag{III.29}$$

Taking into account that

$${}_p\Gamma_k(\alpha n + \beta) = \frac{p^{\frac{\alpha n+\beta}{k}}}{k}\Gamma\left(\frac{\alpha n + \beta}{k}\right), \tag{III.30}$$

the recurrence relation for the classical gamma function and (III.24), it results

$$\left(\frac{\alpha}{k}n + \frac{\beta}{k}\right){}_p\Gamma_k(\alpha n + \beta) = \frac{1}{p}{}_p\Gamma_k(\alpha n + \beta + k). \tag{III.31}$$

Then, the right hand member of (III.29) is equal to

$$p\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}(az^{\frac{\alpha}{k}})^n z^{\frac{\beta}{k}}}{{}_p\Gamma_k(\alpha n+\beta+k)} = pz^{\frac{\beta}{k}}\,{}_p\mathbb{E}_{k,\alpha,\beta+k}(az^{\frac{\alpha}{k}}). \tag{III.32}$$

□

Particular case: If $p = k = \gamma = 1$, it has

$$\int_0^z t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})dt = z^{\beta}E_{\alpha,\beta+1}(at^{\alpha}), \tag{III.33}$$

which coincides with formula (2.3.17) from [18].

Theorem 4. *Let α, β, γ be complex numbers, $k > 0$, $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, $\mathfrak{Re}(\gamma) > 0$, and let ${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z)$ the p-k-Mittag-Leffler function. Then*

$$\mathcal{L}\left\{z^{\frac{\beta}{k}-1}{}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}[\pm(cz)^{\frac{\alpha}{k}}]\right\}(s) = k(sp)^{-\frac{\beta}{k}}\left(1\mp p\left(\frac{c}{ks}\right)^{-\frac{\alpha}{k}}\right)^{-\frac{\gamma}{k}}. \tag{III.34}$$

Proof. Applying the Laplace transform in the right hand member of the (III.1), the relations (II.9) and (II.10), the Laplace transform of the potential function (see e.g. [14], f. (1.4.58)) and the generalized binomial formula given by

$$(1-kw)^{-\frac{\gamma}{k}} = \sum_{k=0}^{\infty}\frac{(\gamma)_{n,k}w^n}{n!}, \tag{III.35}$$

we have

$$\begin{aligned}
\int_0^{\infty} e^{-sz}z^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\,(\pm1)^n(cz)^{\frac{\alpha}{k}n}}{{}_p\Gamma_k(\alpha n+\beta)n!}dz &= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\,(\pm1)^n c^{\frac{\alpha}{k}n}}{{}_p\Gamma_k(\alpha n+\beta)n!}\int_0^{\infty}e^{-sz}z^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}dz\\
&= \frac{k}{(sp)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}}{n!}\left(\frac{c}{ps}\right)^{\frac{\alpha}{k}n}\\
&= \frac{k}{(sp)^{\frac{\beta}{k}}}\sum_{n=0}^{\infty}\frac{(\gamma)_{k,n}}{n!}\left[\left(\frac{c}{ps}\right)^{\frac{\alpha}{k}}\frac{p}{k}\right]^n\\
&= \frac{k}{(sp)^{\frac{\beta}{k}}}\frac{1}{\left[1\mp p\left(\frac{c}{ks}\right)^{\frac{\alpha}{k}}\right]^{\frac{\gamma}{k}}},
\end{aligned}$$

with $\left|p\left(\frac{c}{ks}\right)^{\frac{\alpha}{k}}\right| < 1$. □

Corollary 1. *If $p = k = 1$ in Theorem 4, we have*

$$\begin{aligned}
\mathcal{L}\left\{z^{\beta-1}\,{}_p\mathbb{E}^{\gamma}_{1,\alpha,\beta}(z^{\alpha})\right\}(s) &= \mathcal{L}\left\{z^{\beta-1}E^{\gamma}_{\alpha,\beta}(z^{\alpha})\right\}(s) &\text{(III.36)}\\
&= s^{-\beta}(1-s^{-\alpha})^{-\gamma}, &\text{(III.37)}
\end{aligned}$$

which coincides with formula (11.13) of [13].

Corollary 2. *If $p = k$ in Theorem 4, remembering that ${}_p\Gamma_k(z) = \Gamma_k(z)$, we have that ${}_k\mathbb{E}^{\gamma}_{k,\alpha,b}(z) = E^{\gamma}_{k,\alpha,\beta}(z)$, and*

$$\mathcal{L}\left\{z^{\frac{\beta}{k}-1}\ {}_k\mathbb{E}^{\gamma}_{k,\alpha,b}(z)\right\}(s) = k^{1-\frac{\beta}{k}}s^{-\frac{\beta}{k}}\left(1-k^{1-\frac{\alpha}{k}}s^{-\frac{\alpha}{k}}\right)^{-\frac{\alpha}{k}} \tag{III.38}$$

$$= \frac{k^{1-\frac{\beta}{k}}s^{\frac{\beta\gamma}{k^2}}}{s^{\frac{\alpha}{k}}\left(s^{\frac{\beta}{k}}-k^{1-\frac{\alpha}{k}}\right)^{\frac{\gamma}{k}}}, \tag{III.39}$$

which coincides with formula (II, 29) of [4].

IV The Euler-Beta Transform

The Beta-Euler transform of a given function is defined by the following integral:

$$\mathbf{B}\left(f(z);a;b\right) = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz, \tag{IV.1}$$

with a, $b \in \mathbb{C}$ and $\min\{\mathfrak{Re}(a), \mathfrak{Re}(b)\} > 0$ (see, for example [25]).

Theorem 5. *Let α, β, γ, a and b be complex numbers, $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, $\mathfrak{Re}(\gamma) > 0$, $\mathfrak{Re}(a) > 0$, $\mathfrak{Re}(b) > 0$ and $k > 0$, then*

$$\mathbf{B}\left({}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(z^{\frac{\alpha}{k}}\right);\frac{\beta}{k},\frac{b}{k}\right) = k\ {}_p\Gamma_k(b)\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+b}(\lambda). \tag{IV.2}$$

Proof.

$$\mathbf{B}\left({}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(z^{\frac{\alpha}{k}}\right),\frac{\beta}{k},\frac{b}{k}\right) = \int_0^1 z^{\frac{\beta}{k}-1}(1-z)^{\frac{b}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\lambda^n z^{\frac{\alpha}{k}}n}{{}_p\Gamma_k(\alpha n+\beta)n!}dz, \tag{IV.3}$$

interchanging the order of integration and the summation, and using the relation between the p-k-Gamma function and the classical Gamma function given by (II.11), we have

$$\mathbf{B}\left({}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(z^{\frac{\alpha}{k}}\right),\frac{b}{k},\frac{b}{k}\right) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\lambda^n}{{}_p\Gamma_k(\alpha n+\beta)n!}\int_0^1 z^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}(1-z)^{\frac{b}{k}-1}dz =$$

$$= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\lambda^n}{{}_p\Gamma_k(\alpha n+\beta)n!}\frac{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma\left(\frac{b}{k}\right)}{\Gamma\left(\frac{\alpha n+\beta+b}{k}\right)} = k\ {}_p\Gamma_k(b)\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\lambda^n}{{}_p\Gamma_k(\alpha n+\beta+b)n!} =$$

$$= k\Gamma(b)\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+p}(\lambda). \tag{IV.4}$$

□

Corollary 3. *If $p = k$ in Theorem 5, it result*

$$\mathbf{B}\left(\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(\alpha z^{\frac{\alpha}{k}}\right);\beta,\beta\right) = \Gamma(b)E_{\alpha,\beta+b}(\lambda), \tag{IV.5}$$

which coincides with formula $(II, 37)$ from [4].

Corollary 4. *If $p = k = \gamma = 1$ in Theorem 5, we have*

$$\mathbf{B}\left(E_{\alpha,\beta}\left(\lambda z^{\alpha}\right);\beta;\beta\right) = \Gamma(b)E_{\alpha,\beta+b}(\lambda), \tag{IV.6}$$

which coincides with formula (2.2.14) from [18].

V Some additional properties of ${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z)$.

Theorem 6 (Integral Expresion)**.** *Let α, β and γ be complex numbers, $\alpha > 0$, $\beta > 0$ and $k > 0$, then*

$${}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(z) = p^{\frac{\gamma-\beta}{k}}\frac{1}{{}_p\Gamma_k(\gamma)}\int_0^{\infty} e^{-t}t^{\frac{\gamma}{k}-1}\phi\left(\lambda t;\frac{\alpha}{k};\frac{\beta}{k}\right)dt, \tag{V.1}$$

where $\phi\left(\lambda t;\frac{\alpha}{k};\frac{\beta}{k}\right)$ is the Wright function given by

$$\phi\left(\lambda t;\frac{\alpha}{k};\frac{\beta}{k}\right) = \sum_{n=0}^{\infty}\frac{\lambda^n t^n}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)n!}\ ;\quad \alpha > -1, \beta > 0. \tag{V.2}$$

(See [10] and [9]).

Proof. Starting by the right-hand member of (V.1) and interchanging the order of integration and the summmation we have

$$I = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{1}{{}_p\Gamma_k(\gamma)}\int_0^{\infty}\frac{e^{-t}t^{\frac{\gamma}{k}+n-1}}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}dt = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{1}{{}_p\Gamma_k(\gamma)}\frac{\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}. \tag{V.3}$$

Taking into account (II.11) it result

$$\begin{aligned} I &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}p^{\frac{(\alpha-k)n}{k}}p^{\frac{\beta-\gamma}{k}}\frac{{}_p\Gamma_k(\gamma+nk)}{{}_p\Gamma_k(\gamma)\ {}_p\Gamma_k(\alpha n+\beta)} \\ &= p^{\frac{\beta-\gamma}{k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\left(p^{\frac{\alpha-k}{k}}\lambda\right)^n}{{}_p\Gamma_k(\alpha n+\beta)n!} \\ &= p^{\frac{\beta-\gamma}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(p^{\frac{\alpha-k}{k}}\lambda\right). \end{aligned} \tag{V.4}$$

□

V.1 The p-k-Mittag-Leffler Function and the Riemann-Liouville Fractional Operators

The operators that in this section will consider are those defined by (II.13), (II.14), (II.15) and (II.16).

Theorem 7. *Let α, $b > 0$, $\gamma > 0$, $\nu > 0$, and let $I_{-}^{\frac{\nu}{k}}$ be the right-sided Riemann-Liouville fractional integral. Then we have*

$$I_{-}^{\frac{\alpha}{k}}\left[t^{-\frac{\nu}{k}-\frac{\beta}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha\beta}\left(t^{-\frac{\alpha}{k}}\right)\right](x) = p^{\frac{\nu}{k}}x^{-\frac{\beta}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+\nu}\left(x^{-\frac{\alpha}{k}}\right). \qquad \text{(V.5)}$$

Proof. From (II.11) and (II.12) we find

$$\begin{aligned}\mathfrak{R} &= I_{-}^{\frac{\nu}{k}}\left[t^{-\frac{\nu}{k}-\frac{\beta}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha\beta}\left(t^{-\frac{\alpha}{k}}\right)\right](x)\\ &= \frac{1}{\Gamma\left(\frac{\nu}{k}\right)}\int_x^{\infty} t^{-\frac{\nu}{k}-\frac{\beta}{k}}(t-x)^{\frac{\gamma}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}t^{-\frac{\alpha}{k}n}}{{}_p\Gamma_k(\alpha n+\beta)n!}dt. \qquad \text{(V.6)}\end{aligned}$$

By interchanging the order of integration and summation, we have

$$\mathfrak{R} = \frac{1}{\Gamma\left(\frac{\nu}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}}{{}_p\Gamma_k(\alpha n+\beta)n!}\int_x^{\infty} t^{-\frac{\nu}{k}-\frac{\beta}{k}-\frac{\alpha}{k}}(t-x)^{\frac{\gamma}{k}-1}dt. \qquad \text{(V.7)}$$

Making appropiate changes of variables and using the following integral given by formula 3.191.2 from [11]

$$\int_x^{\infty}(t-x)^{a-1}t^{-c}dt = x^{a-c}\mathbf{B}(a,c-a), \quad \mathfrak{Re}(c) > \mathfrak{Re}(a) > 0; \qquad \text{(V.8)}$$

and $\mathbf{B}(x,y)$ the classical Beta function, can be written

$$\begin{aligned}\mathfrak{R} &= \frac{1}{\Gamma\left(\frac{\nu}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}}{{}_p\Gamma(\alpha n+\beta)n!}x^{\frac{\nu}{k}-\left(\frac{\gamma}{k}+\frac{\beta}{k}+\frac{\alpha n}{k}\right)}\mathbf{B}\left(\frac{\gamma}{k},\frac{\nu+\beta+\alpha n}{k}-\frac{\nu}{k}\right)\\ &= \frac{1}{\Gamma\left(\frac{\nu}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}x^{-\frac{\alpha n}{k}}\Gamma\left(\frac{\beta+\alpha n}{k}\right)}{{}_p\Gamma(\alpha n+\beta)n!\Gamma\left(\frac{\gamma+\beta+\alpha n}{k}\right)}. \qquad \text{(V.9)}\end{aligned}$$

Taking into account (II.11) it results

$$\begin{aligned}\mathfrak{R} &= p^{\frac{\nu}{k}}x^{-\frac{\beta}{k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{k,n}\left(x^{\frac{\alpha}{k}}\right)^n}{{}_p\Gamma_k(\alpha n+\beta+\nu)n!}\\ &= p^{\frac{\nu}{k}}x^{-\frac{\beta}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta+\nu}\left(x^{-\frac{\alpha}{k}}\right). \qquad \text{(V.10)}\end{aligned}$$

□

Corollary 5. *If we take $p = k = 1$ in (V.10), we have*

$$I_{-}^{\nu}\left[t^{-\nu-\beta}E_{\alpha,\beta}^{\gamma}(t^{-\alpha})\right](x) = x^{-\beta}E_{\alpha,\beta+v}^{\gamma}(x^{-\alpha}), \tag{V.11}$$

which coincides with formulae (24) of [23].

Theorem 8. *Let α, β, γ and γ be positive real numbers, and let $\lambda \in \mathbb{R}$. Then there holds*

$$\left(I_{0+}^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}_{k,\alpha\beta}^{\gamma}\left(\lambda t^{\frac{\alpha}{k}}\right)\right]\right)(x) = p^{\frac{\nu}{k}}x^{\frac{\beta+\nu}{k}-1}\ {}_p\mathbb{E}_{k,\alpha,\beta+\nu}^{\gamma}\left(\lambda x^{\frac{\alpha}{k}}\right). \tag{V.12}$$

Proof. Applying the definition of the fractional integral operator, and then by interchanging the order of integration and summation, we have

$$\left(I_{0+}^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}_{k,\alpha\beta}^{\gamma}\left(\lambda t^{\frac{\alpha}{k}}\right)\right]\right)(x) = \frac{1}{\Gamma\left(\frac{\nu}{k}\right)}\int_0^x (x-t)^{\frac{\nu}{k}-1}t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}_{k,\alpha,\beta}^{\gamma}\left(\lambda t^{\frac{\alpha}{k}}\right)dt \tag{V.13}$$

$$= \frac{\alpha}{k}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}\lambda^n}{n!\ {}_p\Gamma_k(\alpha n+\beta)n!}\int_0^x t^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}(x-t)^{\frac{\nu}{k}-1}dt. \tag{V.14}$$

Taking into account that by suitable changes of variables and making use of the Beta function, we have

$$\frac{1}{\Gamma\left(\frac{\gamma}{k}\right)}\int_0^x t^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}(x-t)^{\frac{\gamma}{k}-1}dt =$$

$$= \frac{1}{\Gamma\left(\frac{\gamma}{k}\right)}\int_0^1 x^{\frac{\alpha}{k}-1}(1-\tau)^{\frac{\gamma}{k}-1}x^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}\tau^{\frac{\alpha}{k}n+\frac{\beta}{k}-1}xdt =$$

$$= \frac{a}{\Gamma\left(\frac{\gamma}{k}\right)}x^{\frac{\alpha}{k}n+\frac{\beta}{k}+\frac{\gamma}{k}-1}\frac{\Gamma\left(\frac{\gamma}{k}\right)\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)}{\Gamma\left(\frac{\gamma}{k}+\frac{\alpha}{k}n+\frac{\beta}{k}\right)}. \tag{V.15}$$

By replacing (V.15) in (V.14), it result

$$\left(I_{0+}^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}_{k,\alpha\beta}^{\gamma}\left(\lambda t^{\frac{\alpha}{k}}\right)\right]\right)(x) = x^{\frac{\beta+\nu}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}\lambda^n\Gamma\left(\frac{\alpha}{k}n+\frac{\beta}{k}\right)x^{\frac{\alpha}{k}n}}{{}_p\Gamma_k(\alpha n+\beta)n!}. \tag{V.16}$$

Taking into account (II.11), we put

$$\Gamma\left(\frac{\alpha n+\beta}{k}\right) = \frac{k}{p^{\frac{\alpha n+\beta}{k}}}\ {}_p\Gamma_k(\alpha n+\beta), \tag{V.17}$$

$$\Gamma\left(\frac{\alpha n+\beta+\nu}{k}\right) = \frac{k}{p^{\frac{\alpha n+\beta+\nu}{k}}}\ {}_p\Gamma_k(\alpha n+\beta+\nu). \tag{V.18}$$

From from the above considerations we obtain the thesis.

□

Corollary 6. *If we consider $p = k$, we obtain*

$$\left(I_{0+}^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha\beta}^{\gamma}\left(\lambda t^{\frac{\alpha}{k}}\right)\right]\right)(x) = k^{\frac{\nu}{k}}x^{\frac{\beta+\nu}{k}-1}E_{k,\alpha,\beta+\nu}^{\gamma}\left(\lambda x^{\frac{\alpha}{k}}\right). \tag{V.19}$$

which coincides with (II.19) from [4].

Corollary 7. *If we consider $p = k = 1$, we obtain*

$$\left(I_{0+}^{\nu}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}\left(\lambda t^{\alpha}\right)\right]\right)(x) = x^{\beta+\nu-1}\ {}_{p}\mathbb{E}_{\alpha,\beta+\nu}^{\gamma}\left(\lambda x^{\alpha}\right). \tag{V.20}$$

which coincides with (2.8.11) from [13].

Lemma 3. *For $\lambda \in \mathbb{R}$, these holds the formulae.*

$$\lambda x^{\frac{\alpha}{k}}\ {}_{p}\mathbb{E}_{k,\alpha\beta}^{\gamma}\left(\lambda x^{\frac{\alpha}{x}}\right) = {}_{p}\mathbb{E}_{k,\alpha,\beta-\alpha}^{\gamma}\left(\lambda x^{\frac{\alpha}{k}}\right) - {}_{p}\mathbb{E}_{k,\alpha,\beta-\alpha}^{\gamma-k}\left(\lambda x^{\frac{\alpha}{k}}\right). \tag{V.21}$$

Proof. By the definition (III.1), we have, and taking into account the relation (2.33) from [7],

$${}_{p}(\gamma)_{m-1,k} = \frac{1}{n}\ {}_{p}(\gamma)_{m,k} - \frac{1}{m}\ {}_{p}(\gamma-k)_{m,k}, \tag{V.22}$$

we have

$$\begin{aligned}
\lambda x^{\frac{\alpha}{k}}\ {}_{p}\mathbb{E}_{k,\alpha\beta}^{\gamma}\left(\lambda x^{\frac{\alpha}{k}}\right) &= \sum_{n=0}^{\infty}\frac{{}_{p}(\gamma)_{n,k}\left(\lambda x^{\frac{\alpha}{k}}\right)^{n+1}}{{}_{p}\Gamma_{k}\left(\alpha n+\beta\right)n!} && \text{(V.23)}\\
&= \sum_{m=1}^{\infty}\frac{{}_{p}(\gamma)_{m-1,k}\left(\lambda x^{\frac{\alpha}{k}}\right)^{m}}{{}_{p}\Gamma_{k}\left[\alpha(m-1)+\beta\right](m-1)!} && \text{(V.24)}\\
&= \sum_{m=0}^{\infty}\frac{{}_{p}(\gamma)_{m,k}\left(\lambda x^{\frac{\alpha}{k}}\right)^{m}}{{}_{p}\Gamma_{k}\left(\alpha m-\alpha+\beta\right)m!} - \sum_{m=0}^{\infty}\frac{{}_{p}(\gamma-k)_{m,k}\left(\lambda x^{\frac{\alpha}{k}}\right)^{m}}{{}_{p}\Gamma_{k}\left(\alpha m-\alpha+\beta\right)m!}. && \text{(V.25)}
\end{aligned}$$

Thus, the thesis is obtained. □

Corollary 8. *If we take $\nu = k = 1$ in Lemma 3, we have*

$$\lambda x^{\alpha}\ {}_{1}\mathbb{E}_{1,\alpha\beta}^{\gamma}\left(\alpha x^{\alpha}\right) = \lambda x^{\alpha}E_{\alpha,\beta}^{\gamma}\left(\alpha x^{\alpha}\right) = E_{\alpha,\beta-\alpha}^{\gamma}\left(\alpha x^{\alpha}\right) - E_{\alpha,\beta-\alpha}^{\gamma}\left(\alpha x^{\alpha}\right), \tag{V.26}$$

result that coincies with the formulae (18) due to Saxena an Saigo in [23].

Theorem 9. *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\nu > 0$. Then*

$$I^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_{p}\mathbb{E}_{k,\alpha,\beta}^{\gamma}\left(t^{\frac{\alpha}{k}}\right)\right](x) = p^{\frac{\nu}{k}}x^{\frac{\beta+\nu-\alpha}{k}-1}\left[\ {}_{p}\mathbb{E}_{k,\alpha\beta+\nu-\alpha}\left(x^{\frac{\alpha}{k}}\right) - {}_{p}\mathbb{E}_{k,\alpha,\beta+\gamma-\alpha}\left(x^{\frac{\alpha}{k}}\right)\right]. \tag{V.27}$$

Proof. From (V.12) and (V.21) it result. □

Corollary 9. *If $k = p = 1$ in Theorem 9, we have*

$$I^{\nu}\left[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(t^{\alpha})\right](x) = x^{\beta+\nu-\alpha-1}\left[E_{\alpha,\beta+\ni-\alpha}(x^{\alpha}) - E^{\gamma-1}_{\alpha,\beta+\nu-\alpha}(x^{\alpha})\right], \quad (V.28)$$

which coincides with (2.8.14) from [13] and with formulae (19) from [23].

Theorem 10. *Let $\alpha > 0$, $\beta > 0$,$\gamma > 0$ and $\nu > 0$. Then the Riemann-Liouville fractional derivative of order $\frac{\nu}{k}$ of the auxiliar functions $t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(x^{\frac{\alpha}{k}})$ is given by*

$$D^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(t^{\frac{\alpha}{k}}\right)\right](x) = x^{\frac{\alpha-\nu}{k}-1}p^{-\frac{\gamma}{k}}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta-\nu}(x^{\frac{\alpha}{k}}). \quad (V.29)$$

Proof.

$$D^{\frac{\nu}{k}}\left[t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}})\right](x) = \left(\frac{d}{dx}\right)^m\left[\frac{1}{\Gamma\left(m-\frac{\nu}{k}\right)}\int_0^x t^{\frac{\alpha}{k}n+\frac{\beta}{k}+1}(x-t)^{m-\frac{\nu}{k}-1}dt\right](x)$$

$$= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)n!}\frac{1}{\Gamma\left(m-\frac{\nu}{k}\right)}\left(\frac{d}{dx}\right)^m\int_0^x t^{\frac{\alpha}{k}n+\frac{\beta}{k}+1}(x-t)^{m-\frac{\nu}{k}-1}dt. \quad (V.30)$$

Through a procedure totally analogous to the one realized in Theorem 1 , (V.29) is obtained.

□

Corollary 10. *If $p = k = 1$, we have*

$$D^{\nu}\left[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(z^{\alpha})\right](x) = x^{\beta-1-\nu}E^{\gamma}_{\beta-\nu}(z^{\alpha}), \quad (V.31)$$

which coincides with formulae (2.8.22) from [18], and with (2.1.54) from [14] and with (29) from [23].

VI Aplication

In this section we give a generalization of the kinetic equation involving the p-k-Mittag-Leffler function. The first known generalization of the kinetic equation in the field of Fractional Calculus has been given by Houbold and Mathai in [12] as follows

$$N(t) - N_0 = -c^{\nu}I^{\nu}N(t) \quad (VI.1)$$

where I^{ν} is the fractional integral operator of Riemann-Liouville.

Several authors have proposed and studied generalizations of this equation, for example, Saxena and Kalla [24] considered

$$N(t) - N_0 f(t) = -c^{v} I^{\nu}N(t) \quad (\mathfrak{Re}(v) > 0), \quad (VI.2)$$

where the function f is integrable over $(0, \infty)$.

The kinetic equation has been studied starting from the previous generalization and considering different functions f, in particular, special functions and generalizations of them (see, for example [16], [20], [1], and the references in them)

In this work we study the following generalization of the kinetic equation

$$N(t) - N_0 t^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(-(ct)^{\frac{\alpha}{k}}\right) = -c^{\frac{\alpha}{k}} I^{\alpha}_{k} N(t). \tag{VI.3}$$

which solution is given by the following

Theorem 11. *Let α, β, γ be complex numbers with $\mathfrak{Re}(\alpha) > 0$, $\mathfrak{Re}(\beta) > 0$, $\mathfrak{Re}(\gamma) > 0$; and let $p, k, c \in \mathbb{R}^+ \setminus \{0\}$; then, the solution of VI.3 is given by*

$$N(t) = \frac{c}{k} N_0 \int_0^t H^{1,1}_{1,2}\left[\left(\frac{c}{k}\right)^{\frac{\alpha}{k}} (t-\tau)^{\frac{\alpha}{k}} \,\middle|\, \begin{matrix} (-\frac{\alpha}{k}, 1) \\ (-\frac{\alpha}{k}, 1)\,(0, \frac{\alpha}{k}) \end{matrix}\right] \tau^{\frac{\beta}{k}-1}\ {}_p\mathbb{E}^{\gamma}_{k,\alpha,\beta}\left(-(c\tau)^{\frac{\alpha}{k}}\right) d\tau. \tag{VI.4}$$

Proof. Applying the Laplace transform in the right hand member of the VI.3, using (Laplace transform of the k-Riemann-Liouville fractional integral) and (III.34) we have

$$N(S) = N_0 \frac{s^{\frac{\alpha}{k}}}{s^{\frac{\alpha}{k}} + \left(\frac{c}{k}\right)^{\frac{\alpha}{k}}} \frac{k(sp)^{-\frac{\beta}{k}}}{\left[1 + p\left(\frac{c}{ps}\right)^{\frac{\alpha}{k}}\right]^{\frac{\gamma}{k}}} \tag{VI.5}$$

To reverse the Laplace transform, let us first note that from [17] p. 152 result

$$\frac{s^{\frac{\alpha}{k}}}{s^{\frac{\alpha}{k}} + \left(\frac{c}{k}\right)^{\frac{\alpha}{k}}} = H^{1,1}_{1,1}\left[\left(\frac{sk}{c}\right)^{\frac{\alpha}{k}} \,\middle|\, \begin{matrix} (1,1) \\ (1,1) \end{matrix}\right] \tag{VI.6}$$

Then, by formula 2.21 of [18] we have

$$\mathcal{L}^{-1}\left\{H^{1,1}_{1,1}\left[\left(\frac{sk}{c}\right)^{\frac{\alpha}{k}} \,\middle|\, \begin{matrix} (1,1) \\ (1,1) \end{matrix}\right]\right\}(t) = t^{-1} H^{1,1}_{2,1}\left[\left[\left(\frac{ct}{k}\right)^{\frac{\alpha}{k}}\right]^{-1} \,\middle|\, \begin{matrix} (1,1)\,(0, \frac{\alpha}{k}) \\ (1,1) \end{matrix}\right] \tag{VI.7}$$

Now, by property 1.58 of [18] it turns out that the right hand member of VI.7 is equal to

$$t^{-1} H^{1,1}_{1,2}\left[\left(\frac{ct}{k}\right)^{\frac{\alpha}{k}} \,\middle|\, \begin{matrix} (0,1) \\ (0,1)\ (1, \frac{\alpha}{k}) \end{matrix}\right] \tag{VI.8}$$

Using property 1.60 of [18], the second member of VI.8 is equal to

$$\frac{c}{k} H^{1,1}_{1,2} \left[\left(\frac{ct}{k} \right)^{\frac{\alpha}{k}} \middle| \begin{matrix} \left(-\frac{k}{\alpha}, 1\right) \\ \left(-\frac{k}{\alpha}, 1\right) \ \left(0, \frac{\alpha}{k}\right) \end{matrix} \right] \tag{VI.9}$$

Finally it turns out that

$$\mathcal{L}^{-1} \left\{ \frac{s^{\frac{\alpha}{k}}}{s^{\frac{\alpha}{k}} + \left(\frac{c}{k}\right)^{\frac{\alpha}{k}}} \right\} (t) = \frac{c}{k} H^{1,1}_{1,2} \left[\left(\frac{ct}{k} \right)^{\frac{\alpha}{k}} \middle| \begin{matrix} \left(-\frac{k}{\alpha}, 1\right) \\ \left(-\frac{k}{\alpha}, 1\right) \ \left(0, \frac{\alpha}{k}\right) \end{matrix} \right] \tag{VI.10}$$

and then returning to the equation (VI.5) and using the convolution theorem of Laplace transform has the expected result.

□

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