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OMEGA DERIVATIVE

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Abstract. In this paper, we discuss about the Ω-derivative, a concept which generalizes the classical derivative. Main properties of this generalized derivative are revised. We also study Ω-differential equations and some of its applications.

1. Preliminaries

In the last five decades we have witnessed the development of new differential and integral operators, both fractional and generalized. The latter are defined in terms of an incremental quotient and generate integral operators that may or may not be fractional. To date, the study of this field has attracted the attention of many researchers, not only in pure mathematics, but also in various areas of applied sciences.

Although the 1960s are considered the emergence of generalized local operators (see [13]), before that date some very interesting generalized operators were known.

The following local differentiable operator was previously defined in [5].

Definition 1.1. Let I be an open interval (bounded or unbounded), and let $f : I \to \mathbb{R}$ and $\Omega : I \to \mathbb{R}$ functions such that Ω is continuous and strictly increasing on I. For $x_0 \in I$, function f is termed Ω–differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\Omega(x) - \Omega(x_0)} \tag{1.1}$$

exists. If this limit exists we denote its value by $D_\Omega f(x_0)$, which we call the Ω-derivative of f at x_0.

Remark 1.2. The Ω-derivative generalizes the classical derivative, since for $\Omega(x) = x$, then the Ω–derivative of f is the usual ordinary derivative of f. Also, if $f'(x_0)$ and $\Omega(x_0)$ both exist and $\Omega(x_0) \neq 0$, then

$$D_\Omega f(x_0) = \frac{f'(x_0)}{\Omega'(x_0)}. \tag{1.2}$$

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Additionally if $f(x) = \Omega(x)$, then $D_\Omega f(x_0) = 1$. In constructing a theory of Ω-differential equations, this is perhaps the most serious difficulty to be overcome. Suppose that if $y(x_0) = k \neq 1$, then any Ω-differential equation with initial condition $y(x_0) = 1$ and $f \equiv \Omega$, has no solution, it doesn't matter the nature of the equation itself.

This paper is organized as follows. In Section 2 we gather some concepts and properties regarding the Ω-derivative. In Section 3 we state and prove some results about existence and uniqueness of solutions of Cauchy problems involving Ω-derivatives. In Section 4 the classical model to find the voltage in an RC circuit is studied in the setting of Ω-derivatives. Finally, in Section 5, conclusions and further studies are discussed.

2. First Results

First let us define the class of admissible functions for our theory.

Definition 2.1. Under the same conditions of Definition 1.1, we say that f belongs to the class f_Ω (or admissible) if

$$f_\Omega(I) = \{f : I \to \mathbb{R}, \frac{f'(x)}{\Omega(x)} \neq k, k \in \mathbb{R}\}. \tag{2.1}$$

Following the ideas in [2, 3, 5], we can easily prove the following result.

Theorem 2.2. *Let f and g be admissible Ω-differentiable at a point $t > 0$ and $\alpha \in (0, 1]$. Then*
a) $D_\Omega(af + bg)(t) = aD_\Omega(f)(t) + bD_\Omega(g)(t)$.
b) $D_\Omega(t^p) = \frac{pt^{t-1}}{\Omega'(t)},\ p \in \mathbb{R}$.
c) $D_\Omega(\lambda) = 0,\ \lambda \in \mathbb{R}$.
d) $D_\Omega(fg)(t) = fD_\Omega(g)(t) + gD_\Omega(f)(t)$.
e) $D_\Omega(\frac{f}{g})(t) = \frac{gD_\Omega(f)(t) - fD_\Omega(g)(t)}{g^2(t)}$.

Remark 2.3. One of the most required properties of a derivative operator is the Chain Rule, to calculate the derivative of compound functions, which does not exist in the case of classical fractional derivatives $D_\Omega(f \circ g)(t) = D_\Omega f(g(t)) = f'(g(t))D_\Omega g(t)$.

The following result is easy to obtain (see Theorem 1 in [2]).

Theorem 2.4. *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given admissible function, if f is Ω-derivable in $t_0 \geq 0$, then f is continuous in t_0.*

Definition 2.5. Let f be an admissible function defined on an open interval I. We say that F is an Ω-antiderivative of f on I, if

$$D_\Omega F(x) = f(x), \quad \text{for all } x \in I.$$

We will denote by $R(\Omega)$ the set all Riemann–Stieltjes integrable functions with respect to Ω, where Ω is a continuous, strictly increasing function on a closed, bounded interval $[a, b]$.

Definition 2.6. Let f be an admissible function defined on an open interval I. We say that F is an Ω-antiderivative of f on I, if

$$D_\Omega F(x) = f(x), \quad \text{for all } x \in I.$$

We can define the associate integral (see [5]). Throughout this work, we will suppose that the integral operator kernel Ω, defined below, is of class $R(\Omega)$.

Definition 2.7. Let I be an interval $I \subseteq \mathbb{R}$, $a, t \in I$ and $\alpha \in \mathbb{R}$. The integral operator J_Ω, is defined for every locally integrable function f on I and Ω in the class $R(\Omega)$ as

$$J_{\Omega,a+}(f)(t) = \int_a^t f(s)\Omega'(s)\,ds = \int_a^t f(s)\,d\Omega(s), t > a. \tag{2.2}$$

and ($\Omega'(t)$ a function of constant sign over I)

$$J_\Omega(f)(t) = \int_t^b f(s)\Omega'(s)\,ds = -J_{\Omega,b+}(f)(t),$$

$$J_{\Omega,a}(f)(b) = \int_a^b f(s)\Omega'(s)\,ds = J_{\Omega,b-}(f)(t) + J_{\Omega,a+}(f)(t).$$

Remark 2.8. We can define the function space $L^p_\Omega[a,b]$ as the set of functions over $[a,b]$ such that $(J_{\Omega,a}(f^p)(b)) < +\infty$.

The following two propositions are fundamental, as they relate the previous integral operator with the Ω-derivative. We can think of them as the Fundamental Theorem of Calculus for Ω-derivatives.

Proposition 2.9. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$, and f a Ω-differentiable function on I such that f' is a locally integrable function on I. Then, we have for all $t \in I$*

$$J_{\Omega,a}\big(D_\Omega(f)\big)(t) = f(t) - f(a).$$

Proposition 2.10. *Let I be an interval $I \subseteq \mathbb{R}$, $a \in I$.*

$$D_\Omega\big(J_{\Omega,a}(f)\big)(t) = f(t),$$

for every continuous function f on I and $a, t \in I$.

The following result summarizes some elementary properties of the integral operator $J_{\Omega,a}$.

Theorem 2.11. *Let I be an interval $I \subseteq \mathbb{R}$, $a, b \in I$. Suppose that f, g are locally integrable functions on I, and $k_1, k_2 \in \mathbb{R}$. Then we have*

(1) $J_{\Omega,a}\big(k_1 f + k_2 g\big)(t) = k_1 J_{\Omega,a} f(t) + k_2 J_{\Omega,a} g(t)$,

(2) *if $f \geq g$, then $J_{\Omega,a} f(t) \geq J_{\Omega,a} g(t)$ for every $t \in I$ with $t \geq a$,*

(3) $|J_{\Omega,a} f(t)| \leq J_{\Omega,a} |f|(t)$ *for every $t \in I$ with $t \geq a$.*

There is of course an integration by parts rule in the setting of $J_{\Omega,a}$-integrals and Ω-derivatives. This result is stated as follows.

Theorem 2.12. *(Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ Ω-differentiable functions. Then, the following property holds*

$$J_{\Omega,a}((f)(D_\Omega g(t))) = [f(t)g(t)]_a^b - J_{\Omega,a}((g)(D_\Omega f(t))). \tag{2.3}$$

Improper integrals are defined as follows.

Definition 2.13. Suppose that $f \in R(\Omega)$, for all $b > 0$. Then

$$\int_a^\infty f(s)\, d\Omega(s) := \lim_{b\to\infty} \int_a^b f(s)\, d\Omega(s) \tag{2.4}$$

if the limit exists and it is finite.

For the proof of the Dominated Convergence Theorem (Theorem 2.15), we will need the following lemma.

Lemma 2.14. *If $\int_a^\infty f(s)\, d\Omega(s) < \infty$, then for a given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$\int_b^\infty f(s)\, d\Omega(s) < \varepsilon$$

whenever $b > n_0$.

Proof. If $\int_a^\infty f(s)\, d\Omega(s) = \lim_{b\to\infty} \int_a^b f(s)\, d\Omega(s)$, then for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left| \int_0^\infty f(s)\, d\Omega(s) - \int_0^b f(s)\, d\Omega(s) \right| < \varepsilon \tag{2.5}$$

if $b > n_0$. Since

$$\int_0^\infty f(s)\, d\Omega(s) = \int_0^b f(s)\, d\Omega(s) + \int_b^\infty f(s)\, d\Omega(s). \tag{2.6}$$

Thus

$$\left| \int_b^\infty f(s)\, d\Omega(s) \right| = \left| \int_0^\infty f(s)\, d\Omega(s) - \int_0^b f(s)\, d\Omega(s) \right| < \varepsilon \tag{2.7}$$

if $b > n_0$ and hence

$$\int_b^\infty f(s)\, d\Omega(s) = \left| \int_b^\infty f(s)\, d\Omega(s) \right| < \varepsilon \tag{2.8}$$

with $b > n_0$. $\square$

Now, we state and prove an important result, the Dominated Convergence Theorem.

Theorem 2.15. *Suppose f and f_n $(n = 1, 2, 3, ...)$ are defined on $(0, +\infty)$ and are Ω-integrable on $[t, T]$ whenever $0 < t < T < +\infty$. Also $|f| \leq g$ on $(0, +\infty)$ and $f_n \to f$ uniformly on every compact subset of $(0, +\infty)$ and $\int_0^\infty g(s)\, d\Omega(s) < +\infty$. Then*

$$\lim_{n\to\infty}\int_0^\infty f_n(s)\,d\Omega(s)=\int_0^\infty f(s)\,d\Omega(s). \tag{2.9}$$

Proof. Since $f_n \to f$ uniformly on every compact subset of $(0,+\infty)$, say $[t,T]$, then for any $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\left|\int_0^t f_n(s)\,d\Omega(s)-\int_0^t f(s)\,d\Omega(s)\right|<\frac{\varepsilon}{4}$$

and

$$\left|\int_t^T f_n(s)\,d\Omega(s)-\int_t^T f(s)\,d\Omega(s)\right|<\frac{\varepsilon}{4}$$

if $n\geq n_0$. Now, by Lemma 2.14, we have

$$\begin{aligned}
&\left|\int_0^\infty f_n(s)\,d\Omega(s)-\int_0^\infty f(s)\,d\Omega(s)\right|\\
=\;&\left|\int_0^t f_n(s)\,d\Omega(s)+\int_t^T f_n(s)\,d\Omega(s)+\int_T^\infty f_n(s)\,d\Omega(s)\right.\\
-\;&\left.\left(\int_0^t f(s)\,d\Omega(s)+\int_t^T f(s)\,d\Omega(s)+\int_T^\infty f(s)\,d\Omega(s)\right)\right|\\
\leq\;&\left|\int_0^t f_n(s)\,d\Omega(s)-\int_0^t f(s)\,d\Omega(s)\right|+\left|\int_t^T f_n(s)\,d\Omega(s)-\int_t^T f(s)\,d\Omega(s)\right|\\
+\;&\left|\int_T^\infty f_n(s)\,d\Omega(s)-\int_T^\infty f(s)\,d\Omega(s)\right|\\
<\;&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\int_T^\infty |f_n(s)|\,d\Omega(s)+\int_T^\infty |f(s)|\,d\Omega(s)\\
<\;&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\int_T^\infty g(s)\,d\Omega(s)+\int_T^\infty g(s)\,d\Omega(s)\\
<\;&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+2\frac{\varepsilon}{4}=\varepsilon,
\end{aligned}$$

if $n\geq n_0$. Hence

$$\lim_{n\to\infty}\int_0^\infty f_n(s)\,d\Omega(s)=\int_0^\infty f(s)\,d\Omega(s).$$

□

3. On the Ω-differential equations

The following results will be needed in our work (see chapter 2 in [4]).

Definition 3.1. Let (E,d) be a complete metric space. Mapping A of space E to itself is called contractive mapping, when there exists $q\in[0,1)$ such that $d(A(x),A(y))\leq q\cdot d(x,y)$ for $x,y\in E$.

Theorem 3.2. *(Banach Fixed Point Theorem) Every contracted map A, defined in a complete metric space E, has a unique fixed point, that is, the equation $A(x) = x$, has a unique solution.*

Let us recall the definition of Lipschitz condition.

Definition 3.3. $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition if

$$d\left(f(x_2), f(x_1)\right) \leq K d(x_2, x_1) \tag{3.1}$$

with $K < 1$ and $x_1, x_2 \in I$.

Definition 3.4. An Ω-Cauchy problem is an initial value problem of the form

$$\begin{cases} D_\Omega y(x) = F(x, y(x)) \\ y(x_0) = y_0 \end{cases} \tag{3.2}$$

where $y \in f_\Omega(I)$, $F \in C(I, \mathbb{R})$ and $x_0 \in I$. A solution of the Ω-Cauchy problem is a function y which satisfies (3.2).

Remark 3.5. The above definition means the following:

i. y is a Ω–differentiable function on I.
ii. $(x, y(x))$ is in the domain of definition of F.
iii. $D_\Omega y(x) = F(x, y(x))$ for $x \in I$.

Now, we state and prove our main result, i.e. the existence and uniqueness of the solution of (3.2).

Theorem 3.6. *Under the following conditions:*

1. *$D = I \times \mathbb{R}$.*
2. *$F \in C(I, \mathbb{R})$.*
3. *F satisfies a certain Lipschitz condition on D, $|F(x, y_1(x)) - F(x, y_2(x))| \leq L\,|y_1 - y_2|$.*

the Ω-Cauchy problem (3.2) has a unique solution.

Proof. We will prove that in an interval $|x - x_0| \leq p$ there exists one and only one solution of the problems (3.2). This problem (3.2) is equivalent to the following integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(s, y(s)) d\Omega(t). \tag{3.3}$$

Due to the continuity of F, it is bounded in any closed and bounded set $D' \in D$ which contain the point (x_0, y_0), that is, $|F(x, y(x))| \leq M$. Let us choose the number p in such a way that the following conditions are fulfilled:

1) $(x, y(x)) \in D'$ provided that $|x - x_0| \leq p$ and $|y - y_0| \leq Mp$.
2) $Ld < 1$.

Let us designate by C' the space of the continuous functions φ defined on the segment $|x - x_0| \leq p$ and such that $|\phi(x) - y_0| \leq Mp$, with the metric $d(\varphi_1, \varphi_2) = max_x\,|\varphi(x_1) - \varphi(x_2)|$.
It is very easy to complete the proof, following the integer order case, see for example, [4]. □

We consider now the Ω-Cauchy problem

$$\begin{cases} D_\Omega u(x) = f(x) \\ u(x_0) = \Omega(x_0). \end{cases} \tag{3.4}$$

Readers will have no difficulty in proving that the above problem is equivalent to the following integral equation:

$$u(x) = \Omega(x_0) + \int_{x_0}^{x} f(s)\, d\Omega(s). \tag{3.5}$$

Theorem 3.7. *(A Gronwall-Bellman Inequality) Suppose $u, C \in C(\mathbb{R}, \mathbb{R}^+)$ if for all $t \in [0, T]$*

$$u(t) \leq \Omega(t_0) + \int_{t_0}^{t} C(s)u(s)d\Omega(s), \tag{3.6}$$

then for all $t \in [0, T]$ we have

$$u(t) \leq \Omega(t_0) \cdot \exp\left(\int_{t_0}^{t} C(s)d\Omega(s)\right). \tag{3.7}$$

Proof. . Assuming the hypotheses of the theorem, define $R : [0, T] \to [0, +\infty)$ by $R(t) = \int_{t_0}^{t} C(s)u(s)d\Omega(s)$.

Note that by construction U is Ω-differentiable, and by hypothesis for all $t \in [0, T]$, $u(t) \leq R(t)$.

Differentiating yields that for all $t \in [0, T]$ we have $D_\Omega R(t) = C(t)u(t) \leq C(t)[\Omega(t_0) + R(t)]$, hence for all $t \in [0, T]$ which implies $\frac{dR(t)}{\Omega(t_0)+R(t)} \leq C(t)d\Omega(s)$.

Integrating both sides of the latter inequality

$$\int_{t_0}^{t} \frac{dR(s)}{\Omega(t_0) + R(s)} \leq \int_{t_0}^{t} C(s)d\Omega(s).$$

Hence

$$ln\left(\frac{\Omega(t_0) + R(t)}{\Omega(t_0)}\right) \leq \int_{t_0}^{t} C(s)d\Omega(s)$$

and so

$$u(t) \leq \Omega(t_0)exp\left(\int_{t_0}^{t} C(s)d\Omega(s)\right).$$

□

Recall the definition of the Mittag-Leffler function. For $\alpha, \beta > 0$, it is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}.$$

Definition 3.8. The Ω-exponential function is defined for every $t \geq 0$ by:

$$E_{1,1}(c\mathbb{O}(t)) \tag{3.8}$$

where $c \in \mathbb{R}$ and $\mathbb{O}(t) = J_{\Omega,a+}(1)(t) = \int_0^t d\Omega(s) = \int_0^t \Omega'(s)\, ds$.

It is clear that the following simple identity holds:

$$D_\Omega(E_{1,1}(c\mathbb{O}(t)) = \frac{(E_{1,1}(c\mathbb{O}(t)))'}{\Omega'(t)} = c(E_{1,1}(c\mathbb{O}(t)).$$

From the above, we obtain the following result for the Ω version of linear system:

$$\begin{cases} D_\Omega x(t) = Ax(t), \\ x(0) = x_0 \end{cases} \tag{3.9}$$

where $A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n, t \geq 0$.

Theorem 3.9. *(Existence and Uniqueness). The solution $x(t)$ of Cauchy problem (3.9) exists and it is unique, for all $t \geq t_0 \geq 0$.*

Further we can write the general solution for the system (3.9), with $n = 2$, of the following form:

$$x(t) = C_1 E_{1,1}\left(a\mathbb{O}(t)\right) + C_2 E_{1,1}\left(b\mathbb{O}(t)\right), \tag{3.10}$$

where a and b are eigenvalues of the matrix A and C_1 and C_2 are arbitrary constants. It is easy to verify that the known behavior of the ordinary case is maintained in this frame (overdamped, critical damping and underdamped).

Finally, for a linear Ω-differential equation

$$D_\Omega x(t) + p(t)x(t) = q(t), \tag{3.11}$$

its general solution is given by

$$x(t) = e^{-\int_0^t p(s)\, d\Omega(s)} \left(\int_0^t q(s) e^{\int_0^s p(r)\, d\Omega(r)}\, d\Omega(s) + C \right), \tag{3.12}$$

for all $t \geq t_0 \geq 0$, where C is an arbitrary real constant and p, q are Ω-continuous functions on $[t_0, +\infty)$.

4. An Application

For more than 150 years, electric circuits were a constant source of application problems, which led to the development of new mathematical tools on more than one occasion. Perhaps the iconic example is the Liénard equation. Of course, today we can not ignore these classic problems (see for example [8], [10], [11] and [12]).

In 1864, with only 34 years, James Clerk Maxwell published “A Dynamical Theory of the Electromagnetic Field”, which included and synthesized in their famous equations all the experimental and laws quantitative obtained to the date, this remarkable result was essentially of a mathematical nature, and would not have been possible without the invention of the calculus by Newton and Leibniz. For that reason, although Maxwell published preliminary versions, this work and the theoretical works carried out by Kirchhoff two decades before, mark really the beginning of the fertile relationship between mathematics and “electricity”.

An RC circuit is an electrical circuit composed of resistors and capacitors. The simplest form of RC circuit is the first-order RC circuit, consisting of a resistor and a capacitor, we want to understand the natural response of this circuit, probably the most frequent among all electrical circuits. For a RC circuit, where the capacitor has an individual voltage $V(t_0) = V_0$ the voltage is described in time with the help of the ordinary differential equation (see [9]):

$$\frac{dV}{dt} + \frac{V}{\tau} = \frac{e(t)}{\tau}, \tag{4.1}$$

where $\tau = RC$ is the time constant of the system measured in seconds, R is the resistance measured in Ohm's, C is the capacitance measured in Farads, V(t) is the voltage on the capacitor and e(t) is the source. Taking the initial condition as $V(0) = 0$, and assuming a constant source e_0, the solution of the equation (3.2) is

$$V(t) = e_0\left(1 - e^{-\frac{t}{\tau}}\right). \tag{4.2}$$

In this work we will consider a Ω-version of the equation (4.1), we study the qualitative behavior of the solutions, considering the conformable case and the non conformable one.

To perform a complete analysis, it is necessary to take into account “the distortion” suffered by the derivative when going from the ordinary case to the non-integer case (see [9]). To avoid such “influence” we must consider a parameter based on the kernell of (4.1), with an appropriate dimensionality, that is,

$$\frac{d}{dt} \to \Omega'(\tau) D_\Omega. \tag{4.3}$$

In the above, τ is the fractional time component and it influence is determined by the kernell we used, in this case is $\Omega'(t)$. From properties of Ω-derivative we have

$$\frac{d}{dt} \to \frac{\Omega'(\tau)}{\Omega'(t)}\frac{d}{dt}, \quad 0 < \alpha \leq 1.$$

Instead of (4.1) consider the equation:

$$D_\Omega V(t) + \frac{V(t)}{\Omega'(\tau)} = \frac{e(t)}{\Omega'(\tau)}. \tag{4.4}$$

where $\tau = RC$ is the time constant measured in seconds.

The general solucion of the above equation is

$$V(t) = e^{-\frac{1}{\Omega'(\tau)}\int_0^t d\Omega(s)}\left(\frac{1}{\Omega'(\tau)}\int_0^t e(s)e^{\frac{1}{\Omega'(\tau)}\int_0^s d\Omega(r)}d\Omega(s) + C\right). \tag{4.5}$$

We consider three cases:

I) $\Omega'(t) = t^{\alpha-1} \to 1$ as $\alpha \to 1$, conformable case.

In this case, the solution is given by

$$V(t) = e^{-\tau^{1-\alpha}\int_0^t s^{\alpha-1}} \left(\tau^{1-\alpha} \int_0^t e(s) e^{\tau^{1-\alpha}\int_0^s r^{\alpha-1}dr} s^{\alpha-1} ds + C \right). \tag{4.6}$$

II) $\Omega'(t) = t^\alpha \to t \neq 1$ as $\alpha \to 1$, non-conformable case.

$$V(t) = e^{-\tau^{-\alpha}\int_0^t s^{\alpha}} \left(\tau^{-\alpha} \int_0^t e(s) e^{\tau^{-\alpha}\int_0^s r^{\alpha}dr} s^{\alpha} ds + C \right). \tag{4.7}$$

III) $\Omega'(t) = 1$, i.e., the ordinary case.

On the other hand, if $\Omega'(t) = 1$ for all t, $t_0 = 0$, $V(0) = 0$ and a constant source e_0, the solution of (4.4) is reduced to (4.2).

In the general case II), the solution of equation (4.4), since it is a non conformable model, is not reduced to (4.2).

We will analyze the behavior of the solutions of equation (4.4), in the presence of inputs of different nature.
a) zero (or constant) input $e(t) = k$, $k \in \mathbb{R}$.
b) sinusoidal input.
c) bounded input.

We can consider a much more general model than (4.4):

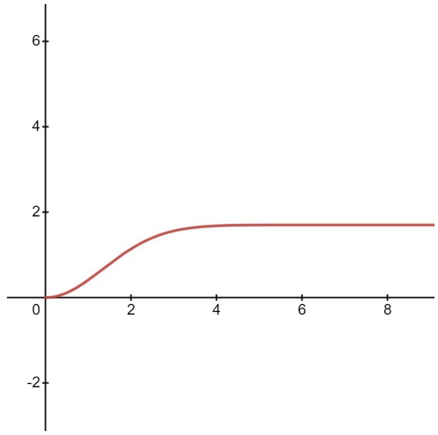
$$D_\Omega V(t) + p(t)\frac{V(t)}{\Omega'(\tau)} = \frac{q(t)}{\Omega'(\tau)}. \tag{4.8}$$

as before $\tau = RC$ is the time constant measured in seconds. The above equation is derived from the ordinary case $V'(t) + p(t)V(t) = q(t)$.

The general solucion of the general equation (4.8) is:

$$V(t) = e^{-\frac{1}{\Omega'(\tau)}\int_0^t p(s)d\Omega(s)} \left(\frac{1}{\Omega'(\tau)} \int_0^t q(s) e^{\frac{1}{\Omega'(\tau)}\int_0^s p(r)d\Omega(r)} d\Omega(s) + C \right). \tag{4.9}$$

The analysis will be done similarly with the three cases (kernels) considered before and the inputs considered before.
For the case in which we have a constant input,

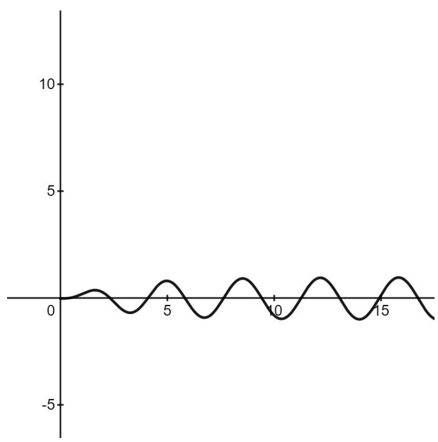


The above graph corresponds to the function

$$\left(\frac{1}{\tau^{(a-1)}}\int_0^x k\cdot\exp\left(\frac{1}{t^{(a-1)}}\cdot\frac{r^a}{a}\right)\cdot r^{(a-1)}dr\right)\exp\left(-\frac{1}{t^{(a-1)}}\cdot\frac{x^a}{a}\right)dt$$

for the values $k = 1.7$, $\tau = 1.8$, $a = 2$.

And, for the case in which the input is sinusoidal, we have



The above graph corresponds to the function

$$\left(\frac{1}{\tau^{(a-1)}}\int_0^x \sin(kr)\cdot\exp\left(\frac{1}{t^{(a-1)}}\cdot\frac{r^a}{a}\right)\cdot r^{(a-1)}dr\right)\exp\left(-\frac{1}{t^{(a-1)}}\cdot\frac{x^a}{a}\right)dt$$

for the values $k = 1.7$, $\tau = 1.8$, $a = 2$.

5. Final Remarks

In this paper we present a new local differential and integral operator with remarkable properties. In particular, the differential operator has shown its strength in simulating various phenomena in which the ordinary derivative cannot provide an answer.

It is clear that the examples shown are only one of the various simulation possibilities offered by this operator, since by varying the kernel Ω and the order α, we have a powerful tool for the study of various local phenomena that only can provide unique modeling with the ordinary derivative, since Ω' and α are fixed and equal to 1.

In the Second Lyapunov Method, the Chain Rule is vital to calculate the derivative of the Lyapunov Function, throughout the solutions of the system under study.

However, we want to advance somewhat in this direction, for this let us consider either the following Generalized Liénard System in the framework of the Ω-derivative:

$$D_\Omega x(t) = y(t) - H(x(t)), D_\Omega y(t) = -g(x(t)), \tag{5.1}$$

as a natural generalization of the classical Liénard system, with $H(x) = \int_{t_0}^{t} h(s)d\Omega(s)$, and f and g are continuous functions such that $h : \mathbb{R} \to \mathbb{R}_+$, and $g : \mathbb{R} \to \mathbb{R}$ with $xg(x) > 0$ for $x \neq 0$. The system (5.1) is equivalent to the equation $(D_\Omega(D_\Omega))x(t) + D_\Omega [H(x(t))] + g(x(t)) = 0$. We consider the following Lyapunov Function

$$V(x, y) = G(x) + \frac{y^2}{2}. \tag{5.2}$$

With $G(x) = \int_{t_0}^{x} g(s)d\Omega(s)$. We calculate the Ω-derivative of (5.2) along the system (5.1):

$$D_\Omega V(x(t), y(t)) = D_\Omega [G(x(t))] + D_\Omega \left[\frac{y^2(t)}{2}\right].$$

$$D_\Omega V(x(t), y(t)) = g(x(t))D_\Omega x(t) + y(t)D_\Omega y(t).$$

From this we have

$$D_\Omega V(x(t), y(t)) = -g(x(t))H(x(t)).$$

Under conditions previously imposed on f and g, we have that V is a positive definite function and its derivative throughout the system (5.1) is non-positive, from this we have the stability according to Lyapunov of the trivial solution of the system (5.1).

Of course, what is presented here does not exhaust the possibilities of applying these operators, for example, we can use the integral operator in the development of new integral inequalities, be it Hermite-Hadamard, Minkowski, Chebishev, Polya-Szego, among others, in a completely new and original frame.

For example, we can state new results on integral inequality of Hermite-Hadamard type, which can be easily proved, following the ideas of [6, 7, 14] inter alia.

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