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A versatile mathematical approach for environmental geomechanic modelling based on stress state decomposition

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ABSTRACT

The main goal of the present paper is to present a mathematical framework for modelling multi-phase non-saturated soil consolidation with pollutant transport based on stress state configurations with special emphasis in its versatility. Non-linear saturation and permeability dependence on suction for both water and pollutant transport is regarded. Furthermore, through the introduction of a suction saturation surface instead of simple suction saturation curves, the implementation of the saturation–suction coupling effect is considerably simplified. The achieved differential equation system is discretized within a Galerkin approach along with the finite element method implementation. A widespread set of practical situations is encompassed by simply setting certain coefficients of the discrete system of equation according to concrete problem conditions. When the model is coped with certain selected fringe conditions, the approach adaptability feature came up showing a robust performance.

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1. Introduction

Mathematical models for solving porous media consolidation stand for an issue widely studied in modern computational mechanics. From the pioneer Biot's work [1] to nowadays complex and robust approaches, a broad range of mechanical situations, thermal conditions, transported fluids, boundary conditions and load types were regarded. The two-phase and three-phase non-saturated cases were undertaken by Ai et al. [2] and Lewis and Schrefler [3] respectively. The non isothermal analysis in saturated models was included in Masters et al. [4], whereas in Yang et al. [5] the non-isothermal case was extended to the unsaturated situation.

Regarding environmental geo-mechanics, different authors have taken on this issue from several standpoints with different aspects or hypothesis under consideration. In Li and Zienkiewicz [6], the fluid transport through porous media in one or two phases separated by an interface with no chemical reactions or components interchange between the phases was presented. In Khoei and Haghghat [7] a solution for material interfaces was brought about independently of element boundaries with an enhanced finite element method (FEM). In Schrefler [8], a mathematical framework assuming a multiphase porous system with voids filled with water, water vapor, dry air and pollutant, relying on Hassanizadeh and Gray averaging theories, see Refs. [9–11], was addressed. Therein and for non isothermal flux, see Khalili and Loret [12], the thermodynamics

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properties of the interface between the different porous media constituents were regarded (Lewis and Schrefler [3]) and, in addition, the relationship between the interface surface tension and the capillary pressure is carefully analyzed.

Furthermore, in Dasgupta et al. [13] the finite difference method was selected for solving a problem of groundwater contamination in a waste-disposal area in Miami, Florida. The relative permeability issue, arising in when wetting and non-wetting phases are regarded, was addressed in Noaman and El-Khatib [14].

Mroginski et al. [15] described a kind of odd relationship that bonds the vertical displacements and the degree of pollutant saturation.

Different degrees of heterogeneity were considered by Ai et al. [16] as well as by Su et al. [17]. In the former, a horizontal multilayered soil with anisotropic permeability undergoing square load was analyzed whilst in the later a first order homogenization on the Representative Volume Element (RVE) was adopted to overcome the heterogeneous issue. In Royer and Boutin [18], the physical interpretation of the three characteristic behaviors of homogenized dual-porosity is evaluated along with memory effects.

Regarding to the numerical solution of the boundary value problem in deformable porous media, different approaches were presented. The coupled consolidation problem of multiphase porous media was discussed in [6,8,15]. Mroginski and Etse [19] propose a finite element (FE) formulation with C1-continuous interpolation functions for non-local saturated porous media. The reference [3] presents a deep study of the numerical solution of coupled consolidation problems in partially saturated soil within the FE framework.

When damage on soil structure is under consideration, [20] assess the consolidation response of a saturated porous media in which structural damage is induced.

Concerning with the constitutive model, Alonso et al. [21] carried out a clay-type partially saturated soils development within the framework of hardening plasticity using two independent sets of stress variables: the excess of total stress over air pressure and the suction. A model based on suction controlled triaxial tests was presented by Sun et al. [22], whereas Graziano and Lancellotta [23] dealt with the derivation of an evolution constitutive equation for deformable porous media.

Khalili et al. [24] presented a mathematical approach for isothermal partially saturated media grounded in a stress state decomposition though regardless of the saturation or the induced matric suction coupling effect. For this issue was subject of large controversial, in Di Rado et al. [25] the evidence of the highly non linear effect that saturation–suction coupling effect renders to the constitutive model and its influence on the symmetry loss in the main system of equations for the isothermal case, were properly settled down. Moreover, in the present manuscript, two advances are brought into consideration with respect to ref [25]: the addition of an immiscible pollutant phase and the introduction of a surface of saturation–suction relationships accounting for the different fluid content instead using a curve for the mixture of fluids. Additionally, a generalization for N immiscible fluid phases is outlined.

This improved mathematical approach, besides spanning a vast sort of isothermal consolidation problems, inherits the three phase model's ductility (see Di Rado et al. [25]) allowing a straightforward reduction to some others problems with a more easy-solving form, namely, the saturated case, non-saturated without suction coupling, consolidation with no presence of pollutant, consolidation with presence of pollutant but without suction coupling, etc. The aforementioned reduction may be carried out by merely setting some coefficients to the required value.

With respect to the mechanical behavior of the soil skeleton (i.e. the constitutive tensor), the same restrictions stated in Ref. [25] hold. One last issue must be underscored. The non-saturated soil consolidation analysis without thermal effects is of great interest for civil constructions like buildings and earth dams, especially when the location area is placed in the north east region of Argentine or south of Paraguay and Brazil. In these locations, many important cities are situated in ancient river's valleys where clay, lime-clay or even sandy soil type with degrees of saturation over 70% (generally due to the groundwater table position) are commonly found. Along with the classical clay-type soil consolidation problem, a brand new situation arises due to the allocation of new industries: the environmental damage. This facts and the possibility of extending the previously developed code were the principal motivations for the present work.

2. The governing equations

2.1. Introduction

For classical mechanic analyses, a continuum distribution of existent particles, either fluid or solid for which the balance laws and constitutive relationships are valid, is frequently accepted. For the case in point, there is an omnipresent phase, i.e., the solid one or solid skeleton, whose voids are taken to be filled with fluid (gas or liquid) separated by a membrane called interface. The difference between the constituents and phases should be emphasized here. On the one hand, the phases are chemically homogeneous portions of the multiphase system which mechanical behavior is assumed to be uniform. On the other hand, the constituents are the individual parts that give rise to the different phases but acting each one independently.

There are two possible levels for describing the multiphase media intergranular configuration: The macroscopic and the microscopic level. At a microscopic level, the actual porous media structure is regarded (see Fig. 1(a)) and, due to this situation, the governing equations are established considering each constituent separately giving rise to a complicated solution, let alone the assessing of microscopic physical and chemical properties. Considering the aforementioned reasons and adding that the microscopic description is generally beyond the civil engineering goals (see Ref. [3]), the macroscopic description is

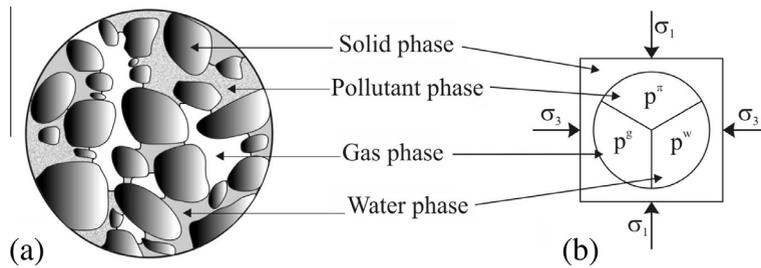


Fig. 1. Porous media description, (a) representative volume element; (b) general stress state.

adopted for the present work. Consequently, when one noteworthy feature of this approach is the fact that, at each material point, all the phases are assumed to be simultaneously present what in turn, renders to a straightforward idealization of the whole structure. In the present paper, the general guidelines originally proposed by Khalili and Khabbaz [24] and later on modified by Di Rado et al. [25] are followed, however, further modifications must be introduced for considering the presence of one or more pollutant phases.

The Fig. 1(b) sketches a representative portion of porous media with its void divided in three parts with each one filled with water, gas (normally air) and pollutant. The outer stress components σ_1 and σ_3 stand for the action of the removed portion of soil, and p^g, p^w and p^π stand for the pore pressures.

For the sake of simplicity, the external stress component σ_2 normal to the paper plane was not drawn albeit considered. In the next paragraph, the stress state indicated in Fig. 1(b) will undergo a convenient decomposition in several different states that, ultimately, give rise to the underlying foundation of the whole approach.

2.2. Solid–fluid relationship. The constitutive equations

The linear momentum balance and mass conservation equations are not enough for reckoning the various pressures acting in the soil portion described in Fig. 1(b). Therefore the adding of solid–fluid constitutive equations, commonly through the widely used effective stress concept (albeit modified for unsaturated soils), is required. Hence, the pore pressures (i.e. water, air and pollutant pressures), the total stress and the effective stress tensor balance leads to the following expression.

$$\sigma'_{ij} = \sigma_{ij} - a_1 p^w \delta_{ij} - a_2 p^\pi \delta_{ij} - a_3 p^g \delta_{ij} \tag{1}$$

being σ'_{ij} the effective stress, p^w, p^g and p^π the pore-water pressure, pore-air pressure and pore-pollutant pressure respectively, and a_1, a_2 and a_3 are the effective stress parameters. In the previous, compressive stresses were regarded positive.

The existence of this basic equation is not merely based in a convenient stress split but it is grounded in porous media thermodynamical investigations (see Diebels and Ehlers [26]). In order to assess the value of parameters a_1, a_2 and a_3 , it is necessary to introduce some specific stress measures of utmost relevance in non saturated soil analysis and widely used in describing its mechanical behavior (see Ref. [3,21,27]). These quantities are the matric water suction, the matric pollutant suction [8,15] and the net normal stress given by

$$p^{cw} = p^g - p^w; \quad p^{c\pi} = p^g - p^\pi; \quad p^n = \bar{\sigma} - p^g \tag{2}$$

with $\bar{\sigma} = (\sigma_1 + \sigma_2 + \sigma_3)/3$ (in principal axes). The Fig. 2 shows an additive decomposition of the stress state with one remarkable aspect: all the aforementioned variables, i.e. $p^{cw}, p^{c\pi}$ and p^n , come up exclusively in one part of the decomposed stress states.

These different stress states, though seemingly arbitrary are selected in order to provide a physical meaning to the sought effective parameters. State (1) stands for a soil portion with a pore pressure equal to the water pore pressure and with an external isotropic pressure set to the same value. States (4) and (5) are comparable to state (1) but shifting the pressure values. State (2) stands for a sample with a null water pore pressure and with a pollutant pore, an air pore and an external isotropic pressures equal to the water suction, namely $p^{cw} = p^g - p^w$. In State (3), the situation is similar to State (2) because the water pore pressure swaps its contribution with the pollutant pore pressure bringing in the pollutant suction $p^{c\pi} = p^g - p^\pi$. States (6) shows a null pore pressure and an external isotropic pressure $\bar{\sigma} - p^g$ whereas State (7) exhibits the same absence of pore pressure but with external stress components $\sigma_i - \bar{\sigma}$. The different components depicted hereinafter, allows writing the volumetric strain of the soil portion in the following fashion:

$$\epsilon_{ii} = \epsilon_{ii}^{(1)} + \epsilon_{ii}^{(2)} + \epsilon_{ii}^{(3)} + \epsilon_{ii}^{(4)} - \epsilon_{ii}^{(5)} + \epsilon_{ii}^{(6)} + \epsilon_{ii}^{(7)} \tag{3}$$

being

$$\begin{aligned} \epsilon_{ii}^{(1)} &= c_s p^w & \epsilon_{ii}^{(2)} &= c_{mw} (p^g - p^w) & \epsilon_{ii}^{(3)} &= c_{m\pi} (p^g - p^\pi) & \epsilon_{ii}^{(4)} &= c_s p^\pi \\ \epsilon_{ii}^{(5)} &= c_s p^g & \epsilon_{ii}^{(6)} &= c (\bar{\sigma} - p^g) & \epsilon_{ii}^{(7)} &= 0 \end{aligned} \tag{4}$$

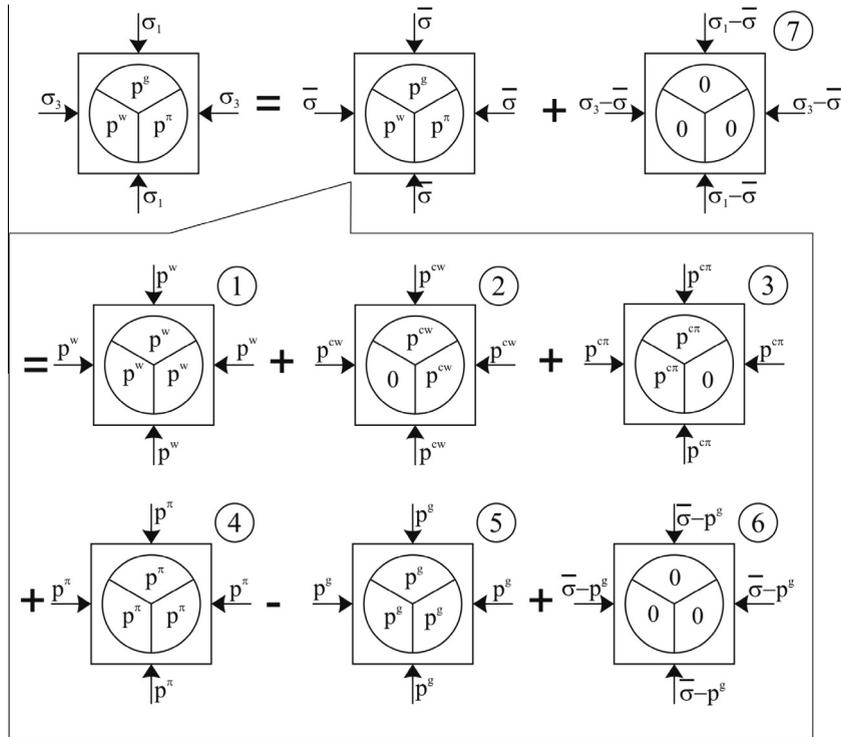


Fig. 2. Decomposition.

In Eq. (4), $c_{mw} = 1/K_{mw}$ is the compressibility of the solid structure with respect to a change in water suction p^w , and $c_{m\pi} = 1/K_{m\pi}$ is the compressibility of the soil structure with respect to a change in matric pollutant suction $p^{c\pi}$ (see Pao and Lewis [28]). For reckoning the quantities K_{mw} and $K_{m\pi}$ the following expressions are suggested:

$$K_{mw} = \frac{K_T K_s}{S_w K_s + \frac{S_w}{S_f} S_g K_T + \frac{S_w}{S_f} K_T}; \quad K_{m\pi} = \frac{K_T K_s}{S_\pi K_s + \frac{S_\pi}{S_f} S_g K_T + \frac{S_\pi}{S_f} K_T} \quad (5)$$

in which S_w, S_π, S_g are the water, pollutant and gas (air) fluid saturation degree respectively, and $S_l = S_w + S_\pi$ is the liquid fluid saturation degree. The different types of saturation may be obtained by:

$$S_w = \frac{V_w}{V_v}, \quad S_\pi = \frac{V_\pi}{V_v} \quad \text{and} \quad S_g = \frac{V_g}{V_v} \quad (6)$$

being V_w, V_π, V_g and V_v the pore-water volume, the pore-pollutant volume, the pore-gas volume and the total void volume respectively, bearing in mind that the identity $V_v = V_w + V_\pi + V_g$, holds.

The other quantities involved in Eq. (4) are: $c_s = \frac{1}{K_s}$, the compressibility of the soil grains; $c = \frac{1}{K_T}$, the drained compressibility of the soil structure; $K_T = (1 - \alpha)K_s$, the bulk modulus of the overall skeleton (see Ref. [3]) in which “ α ” is the Biot constant (see Ref. [1]). Substituting Eq. (4) in Eq. (3) and rearranging terms, yields

$$\varepsilon_{ii} = c_s p^w + c_{mw} (p^g - p^w) + c_{m\pi} (p^g - p^\pi) + c_s p^\pi - c_s p^g + c(\bar{\sigma} - p^g) + 0 \quad (7)$$

Whereas the deviatoric tensor is exactly the same for total and effective stresses, Eq. (1) may be rephrased in terms of volumetric stresses straightforwardly:

$$\bar{\sigma}' = (\bar{\sigma} - a_1 p^w - a_2 p^g - a_3 p^\pi) \quad (8)$$

Taking the product with the drain compressibility of the soil structure in both members of the previous equation, lead to:

$$c\bar{\sigma}' = c(\bar{\sigma} - a_1 p^w - a_2 p^g - a_3 p^\pi) \quad (9)$$

The first member involves the product between soil structure volumetric stress (effective) and the drained compressibility giving the total volumetric strain, then:

$$\varepsilon_{ii} = c(\bar{\sigma} - a_1 p^w - a_2 p^g - a_3 p^\pi) \quad (10)$$

Comparing Eq. (7) with Eq. (10), the effective stress parameters a_1 , a_2 and a_3 may be re-interpreted as functions of the soil parameters, and they are given by:

$$\begin{aligned} a_1 &= \frac{c_{mw} - c_s}{c} \\ a_2 &= \frac{c_{m\pi} - c_s}{c} \\ a_3 &= 1 + \frac{c_s - c_{mw} - c_{m\pi}}{c} \end{aligned} \quad (11)$$

Then, expressing the compressibility coefficients c_s and c in terms of K_s and K_T , respectively, as well as c_{mw} according to Eq. (5), a simple connection between the effective stress parameters above reached and the well known Biots coefficient, α , may be attained (for the sake of brevity, algebraic manipulation were omitted):

$$\begin{aligned} a_1 &= \frac{c_{mw} - c_s}{c} = \alpha S_w \\ a_2 &= \frac{c_{m\pi} - c_s}{c} = \alpha S_\pi \\ a_3 &= 1 + \frac{c_s - c_{mw} - c_{m\pi}}{c} = \alpha S_g \end{aligned} \quad (12)$$

In order to render the approach more familiar to soil mechanics experts, the relationships put forward in Eq. (12) will be forwarded whenever it is possible.

2.3. Mechanical equilibrium

In order to extend the conclusions arisen in the previous paragraph to the differential equation for lineal momentum balance, it is crucial to consider that Fig. 1 sketches a small portion of the soil skeleton. Taking time derivative of Eq. (1) however disregarding the time derivatives of coefficients a_1 , a_2 and a_3 , the following equation is obtained:

$$\dot{\sigma}'_{ij} = \dot{\sigma}_{ij} - (a_1 \dot{p}^w + a_2 \dot{p}^\pi + a_3 \dot{p}^g) \quad (13)$$

In the previous, the coefficients a_1 , a_2 and a_3 were taken as tangent modulus and in consequence actualized after each time iteration. Now, the introduction of Eq. (12) in Eq. (14), leads to the pursued solid–fluid constitutive expression.

$$\dot{\sigma}'_{ij} = \dot{\sigma}_{ij} - \alpha \delta_{ij} (S_w \dot{p}^w + S_\pi \dot{p}^\pi + S_g \dot{p}^g) \quad (14)$$

The classical quasistatic linear momentum balance is given by

$$(\dot{\sigma}'_{ij})_j + \dot{F}_i = 0 \quad (15)$$

being F_i the components of the total body force. Replacing Eq. (14) in Eq. (15):

$$(C_{ijkl} \dot{\epsilon}_{kl})_j + a_1 \dot{p}^w_i + a_2 \dot{p}^\pi_i + a_3 \dot{p}^g_i + \dot{F}_i = 0 \quad (16)$$

where, C_{ijkl} is the tangent constitutive tensor and $\dot{\epsilon}_{kl}$ is the rate of linear deformation tensor.

2.4. Liquid flux model

To obtain the law that rules the fluids flow through porous media, the next equations are deemed (see [3,27]):

1. Momentum balance of the fluids constituents regarding the interaction forces and the quasi-static situation:

$$\sigma^l_{ij,j} + n_i \rho^l b_i + \hat{p}^l_i = 0 \quad (17)$$

In the above equation, σ^l_{ij} is the fluid stress tensor, n_i is the fluid porosity, ρ^l is the fluid effective density, b_i is the body force and \hat{p}^l_i is the unit local interaction force between constituents.

2. Constitutive equations for fluid phase.

$$\sigma^l_{ij} = -n_i p^l \delta_{ij} + \sigma^l_{ij} \quad (18)$$

$$\hat{p}^l_i = p^l (n_i)_{,i} + \hat{p}^l_i \quad (19)$$

Eq. (18) follows the same guidelines stated for Eq. (1), being σ^l_{ij} the stress tensor due to frictional fluid properties. In Eq. (19), \hat{p}^l_i is the viscous drag force. Depending on the given typical length of the solid–fluid problem, either \hat{p}^l_i or σ^l_{ij} are dominant and the other negligible (see Ehlers et al. [29]): for the case in point, the following expressions are obtained:

$$\sigma_{ij}^l = 0 \quad (20)$$

$$\hat{p}_i^l = -n_l \mu^l K_{ij}^{-1} v_j^l \quad (21)$$

Above, μ^l is the relative effective viscosity, K_{ij} is the intrinsic anisotropic permeability tensor and v_j^l is the fluid discharge velocity. For the presence of more than one fluid is considered in the present model, the permeability tensor must be modified in order to reflect the fluids interaction (see Fetter [30]). Mostly, this interaction is brought about introducing a dimensionless coefficient, i.e. the relative permeability that may be assessed following any of the many proposed models (see Corey [31]). Replacing Eq. (18) in Eq. (17) and considering Eqs. (20) and (21) as well as the identity $(n_l p^l \delta_{ij})_{,i} = n_l p_{,i}^l \delta_{ij} + p^l n_{l,i} \delta_{ij}$, the well-known Darcys law is attained however modified by the presence of more than one fluid, is given by:

$$v_i^l = -\frac{k_{ij} k^l}{\gamma^l} (p_{,j}^l - \rho^l b_j) \quad (22)$$

In Eq. (22), the relationship $K_{ij}^l = (\mu^l / \gamma^l) k_{ij} k^l$ with γ^l being the fluid specific weight, k_{ij} is the Darcys permeability coefficient and k^l is the relative permeability, was regarded. This expression holds providing that the permeability tensor is in principal axes, namely $K_{ij}^l = 0$ if $i \neq j$.

3. Fluid mass balance:

$$-(\rho^l v_i^l)_{,i} = \frac{\partial}{\partial t} (n_l \rho^l) \quad (23)$$

Replacing Eq. (22) in Eq. (23) and disregarding the effect of mass forces, leads to (see Ref. [24])

$$-\frac{1}{\rho^l} \left(\rho^l \frac{K_{ij}^l}{\mu^l} p_{,j}^l \right)_{,i} = -n_l c_l \dot{p}^l + \frac{1}{V} \dot{V}_l \quad (24)$$

Herein, V is the total volume and c_l is the fluid compressibility. In Eqs. (17)–(24), l stands for any of the present immiscible liquid fluids, aqueous and non aqueous. If water is under consideration, l will be replaced by w and in the case of pollutant, a π symbol will be used.

2.5. Air flux model

The Fick's law is commonly used to describe the air flow through unsaturated soils. According to this law, the rate of mass transfer for a diffusing substance across a unit area (J_{gi}) is proportional to the concentration gradient of the diffusing substance (C) [32]. This definition renders the following expression:

$$J_{gi} = -D_{ij}^* C_{,j} \quad (25)$$

where D_{ij}^* is a function of the volume-mass properties, the air density and the diffusion coefficient. For isothermal conditions and satisfying the conservation of air mass, using a similar procedure than the one employed for the water phase, the following expression is obtained (see Ref. [24]):

$$-\frac{1}{\rho^g} \left(D_{ij}^* p_{,j}^g \right)_{,i} = -\frac{n_g}{P} \dot{p}^g + \frac{1}{V} \dot{V}_g \quad (26)$$

with P being the absolute pressure and $n_g = V_g / V$ stands for the relative air porosity and V is the total volume.

2.6. Relating variables by stress states consideration

Eqs. (24) and (26) are the governing differential equations describing flow of liquids and air through unsaturated porous media, respectively. These, along with Eq. (16), denote an apparently under-determined system of equations, namely, the unknowns p^l , p^g , σ_{ij} , \dot{V}_l and \dot{V}_g exceeds the number of independent equations, requiring hence relating the volume variables with the displacement and pressures unknowns (ultimately the primary unknowns).

In the upcoming paragraph, a stress state-based relationship as well as the addition of an adequate constitutive equation in terms of the displacement field will be addressed. Hence Eqs. (16), (24) and (26) could be solved simultaneously.

2.6.1. Scheme 1

The state (1) of Fig. 3 corresponds to an external isotropic pressure, $d\bar{\sigma}$, an internal pore-water pressure dp^w , an internal pore-pollutant pressure dp^π and an internal pore-air pressure dp^g . State number (2) shows a tantamount internal (i.e. water, air and pollutant pores) and external pressure, all set to water pore pressure value. Finally, state (3) stands for a portion of soil alike the precedent but with water pore pressure set to zero. All aforementioned idealized stress states, allow to apply

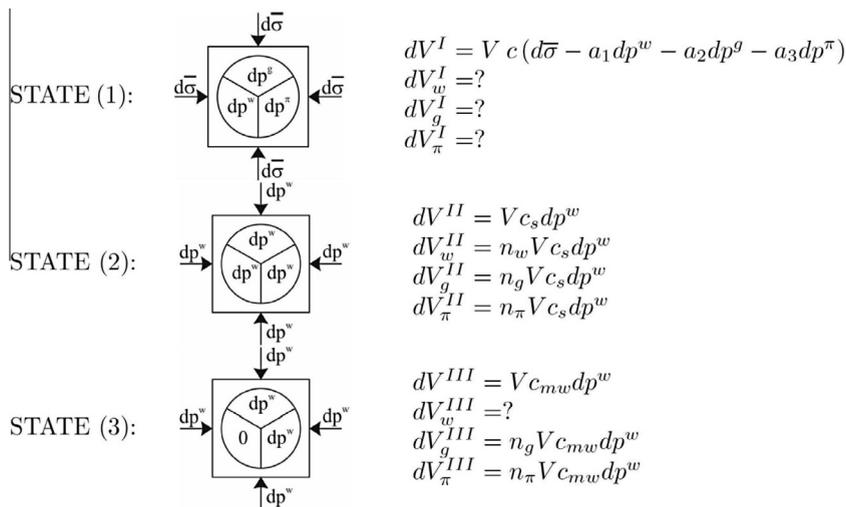


Fig. 3. Set of soil states required for relating V_w . All differential volumes are assessed by recurrent application of Eq. (10).

Betty's reciprocal law to the states [(1)-(2)], [(1)-(3)], and [(2)-(3)]. This valuable mathematical tool lets unveiling the hidden connections between unknowns (the one relating V_w). Accordingly:

$$(dV - dV_w - dV_g - dV_\pi) dp^w = V c_s dp^w d\bar{\sigma} - n_w V c_s dp^w dp^w - n_g V c_s dp^w dp^g - n_\pi V c_s dp^w dp^\pi \tag{27}$$

$$(dV - dV_g - dV_\pi) dp^w = V c_{mw} dp^w d\bar{\sigma} - n_g V c_{mw} dp^w dp^g - dV_w^{III} dp^w - dV_\pi^{III} dp^\pi \tag{28}$$

$$V c_{mw} dp^w dp^w - n_g V c_{mw} dp^w dp^w - dV_w^{III} dp^w - dV_\pi^{III} dp^\pi = V c_s dp^w dp^w - n_\pi V c_s dp^w dp^w - n_g V c_s dp^w dp^w \tag{29}$$

Subtracting Eqs. (28) and (29) from Eq. (27), gives:

$$\frac{dV_w}{V} = (c_{mw} - c_s) d\bar{\sigma} - [c_{mw}(1 - n_g) + c_s(n_\pi - 1 - n_w + n_g) - n_\pi c_{mw}] dp^w - n_g(c_{mw} - c_s) dp^g - n_\pi(c_{mw} - c_s) dp^\pi \tag{30}$$

Considering the differential form of Eq. (10), introducing $n = n_g/S_g = n_w/S_w = n_\pi/S_\pi$ and after some algebraic manipulation, Eq. (30) becomes:

$$\begin{aligned} \frac{dV_w}{V} = & \alpha S_w d\varepsilon_{ii} - \left[(S_g + S_\pi)(\alpha - n) \left(\frac{1}{K_{mw}} - \frac{1}{K_s} \right) + (\alpha - n) \frac{S_w}{K_s} \right] dp^w + \left[S_g(\alpha - n) \left(\frac{1}{K_{mw}} - \frac{1}{K_s} \right) \right] dp^g \\ & + \left[S_\pi(\alpha - n) \left(\frac{1}{K_{mw}} - \frac{1}{K_s} \right) \right] dp^\pi \end{aligned} \tag{31}$$

Substituting Eq. (31) in Eq. (24) with l equal to water, taking the time derivative and considering the chain rule $\dot{S}_w = \frac{dS_w}{dp^{cw}} \dot{p}^{cw}$, $\dot{S}_\pi = \frac{dS_\pi}{dp^{c\pi}} \dot{p}^{c\pi}$ along with some algebraic elaboration, leads to:

$$\begin{aligned} & \frac{1}{\rho^w} \left(\rho^w \frac{K_{ij}^w}{\mu^w} p_j^w \right)_i + \alpha S_w \dot{\varepsilon}_{ii} - S_w \left[\frac{n}{K_w} + \frac{(\alpha - n)}{K_s} \right] \left(S_w - \frac{C_{sw}}{n} dp^w + \frac{C_{sw}}{n} dp^g + \frac{C_{s\pi}}{n} dp^g - \frac{C_{s\pi}}{n} dp^\pi \right) \\ & - \frac{(\alpha - n)}{K_T} \left(- \frac{C_{sw}}{n} dp^w + (S_w - 1) + \frac{C_{sw}}{n} dp^g + \frac{C_{s\pi}}{n} dp^g - \frac{C_{s\pi}}{n} dp^\pi \right) \dot{p}^w \\ & - S_w \left[\frac{(\alpha - n)}{K_s} \left(S_g - \frac{C_{sw}}{n} dp^g + \frac{C_{sw}}{n} dp^w - \frac{C_{s\pi}}{n} dp^g + \frac{C_{s\pi}}{n} dp^\pi \right) \right. \\ & \left. - \frac{(\alpha - n)}{K_T} \left(\frac{C_{sw}}{n} dp^w - \frac{C_{sw}}{n} dp^g + S_g - \frac{C_{s\pi}}{n} dp^g + \frac{C_{s\pi}}{n} dp^\pi \right) \right] \dot{p}^g - S_w \left[\frac{(\alpha - n)}{K_s} (S_\pi) - \frac{(\alpha - n)}{K_T} (S_\pi) \right] \dot{p}^\pi = 0 \end{aligned} \tag{32}$$

in the above,

$$C_{sw} = n \frac{dS_w}{dp^{cw}}; \quad C_{s\pi} = n \frac{dS_\pi}{dp^{c\pi}} \tag{33}$$

Setting out these coefficients implies further consideration which will be accounted for in Section 2.7.1.

2.6.2. Scheme 2

Likewise the preceding section, the following combination of states is regarded towards finding relationships involving V_π (see Figs. 4, 5).

Applying Betty's reciprocal law to the states [(1)–(5)], [(4)–(5)] and [(1)–(4)], the subsequent relationships are addressed:

$$\begin{aligned} & \frac{1}{\rho^\pi} \left(\rho^\pi \frac{K_{ij}^\pi}{\mu^\pi} p_j^\pi \right)_i + \alpha S_\pi \dot{\epsilon}_{ii} - S_\pi \left[\frac{n}{K_\pi} + \frac{(\alpha - n)}{K_s} \left(S_\pi - \frac{C_{s\pi}}{n} dp^\pi + \frac{C_{s\pi}}{n} dp^g + \frac{C_{sw}}{n} dp^g - \frac{C_{sw}}{n} dp^w \right) \right. \\ & \quad \left. - \frac{(\alpha - n)}{K_T} \left(-\frac{C_{s\pi}}{n} dp^\pi + (S_\pi - 1) + \frac{C_{s\pi}}{n} dp^g + \frac{C_{sw}}{n} dp^g - \frac{C_{sw}}{n} dp^w \right) \right] \dot{p}^\pi \\ & \quad - S_\pi \left[\frac{(\alpha - n)}{K_s} \left(S_g - \frac{C_{s\pi}}{n} dp^g + \frac{C_{s\pi}}{n} dp^\pi - \frac{C_{sw}}{n} dp^g + \frac{C_{sw}}{n} dp^w \right) \right. \\ & \quad \left. - \frac{(\alpha - n)}{K_T} \left(\frac{C_{s\pi}}{n} dp^\pi - \frac{C_{s\pi}}{n} dp^g + S_g - \frac{C_{sw}}{n} dp^g + \frac{C_{sw}}{n} dp^w \right) \right] \dot{p}^g - S_\pi \left[\frac{(\alpha - n)}{K_s} S_w - \frac{(\alpha - n)}{K_T} S_w \right] \dot{p}^w = 0 \end{aligned} \tag{34}$$

2.6.3. Scheme 3

The next combination of states was selected towards finding the link involving V_g .

As can be viewed, in state (6) all pore pressures and the external pressure are equal to dp^g . Meanwhile, in State (7), the gas pressure is set to zero being this fact the only difference with regards to the previous state. Applying Betty's reciprocal law to the states [(1)–(6)], [(6)–(7)] and [(1)–(7)], gives:

$$\frac{dV_g}{V} = S_g \left[\alpha \epsilon_{ii} - \frac{(\alpha - n)}{K_s} (S_w dp^w S_\pi dp^\pi + S_g dp^g) - \frac{(\alpha - n)}{K_T} (S_w dp^{cw} + S_\pi dp^{c\pi}) \right] \tag{35}$$

Substituting Eq. (35) in Eq. (26), taking time derivative, and carrying out similar modifications than those performed in the other fluid phase, yields:

$$\begin{aligned} & \frac{1}{\rho^g} \left(D_{ij}^* p_j^g \right)_i - \left[\frac{(\alpha - n)}{K_s} S_g \left(S_w + \frac{C_{sw}}{n} (dp^g - dp^w) \right) + \frac{(\alpha - n)}{K_T} S_g \left(-S_w - \frac{C_{sw}}{n} (dp^g - dp^w) \right) \right] \dot{p}^w \\ & \quad - \left[\frac{(\alpha - n)}{K_s} S_g \left(S_\pi + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) + \frac{(\alpha - n)}{K_T} S_g \left(-S_\pi - \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) \right] \dot{p}^\pi \\ & \quad - \left[\frac{n S_g}{P} + \frac{(\alpha - n)}{K_s} S_g \left(S_g - \frac{C_{sw}}{n} (dp^g - dp^w) - \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) \right. \\ & \quad \left. + \frac{(\alpha - n)}{K_T} S_g \left(S_w + S_\pi + \frac{C_{sw}}{n} (dp^g - dp^w) + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) \right] \dot{p}^g = 0 \end{aligned} \tag{36}$$

A further revision of Eqs. (32), (34) and (36), leads to the acknowledgment that all of them as well as Eq. (16), rely on the primary unknowns same variables. Thus, the seemingly under-determined system becomes now a determined system.

2.7. Coupling of the mechanical equilibrium with flux

Gathering Eqs. (16), (32), (34) and (36), and introducing a group of parameters with the aim of turning the system lighter and easy-to-handle, it is rendered:

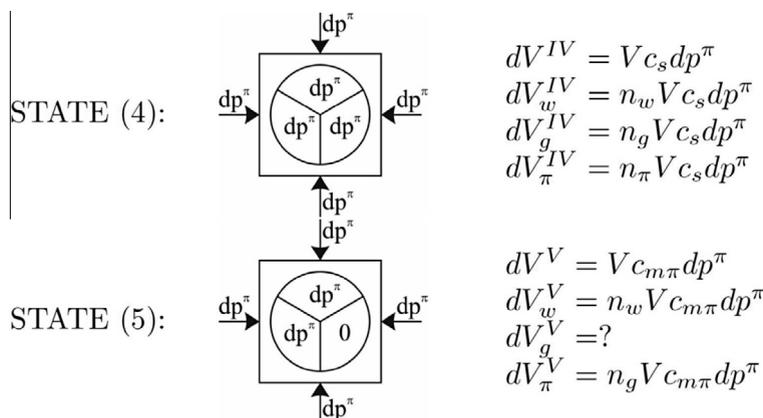


Fig. 4. Set of soil states required for relating V_π .

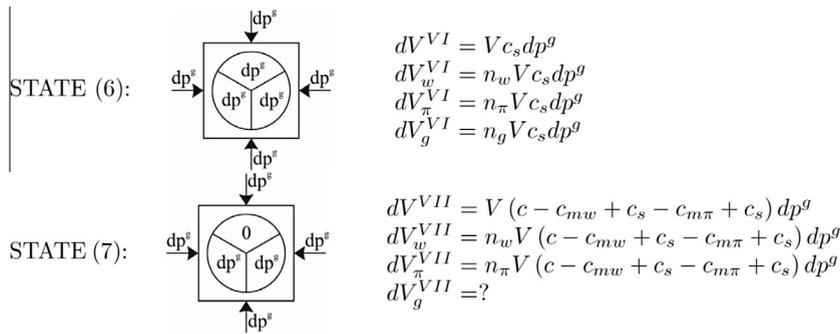


Fig. 5. Set of soil states required for relating V_g .

$$\begin{aligned}
 (C_{ijkl} \dot{\epsilon}_{kl})_j + a_1 \dot{p}_i^w + a_2 \dot{p}_i^\pi + a_3 \dot{p}_i^g + \dot{F}_i &= 0 \\
 a_1 \dot{\epsilon}_{ii} - a_{11} \dot{p}^w - a_{12} \dot{p}^\pi - a_{13} \dot{p}^g + \frac{1}{\rho^w} \left(\rho^w \frac{K_{ij}^w}{\mu^w} p_j^w \right)_i &= 0 \\
 a_2 \dot{\epsilon}_{ii} - a_{21} \dot{p}^w - a_{22} \dot{p}^\pi - a_{23} \dot{p}^g + \frac{1}{\rho^\pi} \left(\rho^\pi \frac{K_{ij}^\pi}{\mu^\pi} p_j^\pi \right)_i &= 0 \\
 a_3 \dot{\epsilon}_{ii} - a_{31} \dot{p}^w - a_{32} \dot{p}^\pi - a_{33} \dot{p}^g + \frac{1}{\rho^g} \left(D_{ij}^* p_j^g \right)_i &= 0
 \end{aligned} \tag{37}$$

All the coefficients involved in Eq. (37) are written out in A.

Eqs. (37) stand for a system of partial differential equations for the solution of soil consolidation problems with pollutant transport derived out of a combination of a collection of stress situations applied to a soil system. This formulation leads to non symmetrical matrices when the finite elements method is applied and may be straightforward reduced to any other situations therein encompassed (i.e., the saturated case, non-saturated without pollutant, etc.) by manually adjusting selected coefficients (regarding on the degree of saturation of each phase). The simplicity of this fact rules out any possibility of conceptual drawbacks at the time that increases computer resources optimization. However, some of the coefficients require a more detailed inspection, e.g. C_{sw} and $C_{s\pi}$.

2.7.1. Suction–saturation and relative permeability coefficients

Both C_{sw} and $C_{s\pi}$ involve a derivative of the water saturation and the pollutant saturation with respect to its respective suction. Therefore, functions relating, on the one hand, S_w with p^{cw} and, on the other hand, S_π with $p^{c\pi}$ are required. Di Rado et al. [25] presents a general guidelines on how to cope with these kind of functions towards the C_{sw} determination. Therein, saturation–suction curves (SSC) gave the experimental support to the mathematical approach (see Fredlund and Xing [32]). However, those curves were sketched out when the only present fluid was water. When the soil structure is permeated with pollutants, the interactions between both fluids and the air phase, bring forth a family of pollutant SSC for each state of water saturation and conversely (see Mroginski et al. [15] and Arega and Hayter [33]). Along with this family of functions, the relative permeability coefficient must be evaluated.

An effective strategy was brought about in Hicks and Grader [34] in which, resting on reduced saturation spaces obtained through experimental testes in which one partial saturation is held constant whilst the others vary, the relative permeability coefficients for the three-phase system are assessed. Regarding the suction saturation curves for the aforementioned system, a tactic somehow analogous to Hicks and Grader [34] may be addressed here by redesigning the experimental test underlying the ternary diagrams:

- (a) C_{sw} : Maintaining the same basic variables for the plane triangle, namely the different degree of saturation of each phase, a different additional dimension is regarded, viz. the water matric suction. The experimental test must be conducted setting the pollutant saturation to constant values and therefore evaluating the suction changes with water saturation (see Fig. 6(a)). With the SSC hereby obtained and following similar guidelines to those given in Ref. [32], C_{sw} may be reckoned. Furthermore, once C_{sw} is evaluated for several points on segments parallel to the gas saturation side of the basic triangle (i.e., lines of constant gas saturation), isolines may be sketched rendering a C_{sw} map, admitting that some suitable mathematical model is adjusted (see Fig. 6(a)).
- (b) $C_{s\pi}$: This coefficient may be attained in a process tantamount to the previous one with a single however fundamental shift: The laboratory tests must be managed holding the water saturation unaltered, see Fig. 6(b) (see Refs. [35] [36,37]).

3. Multiphase flow. Generalization to N mutually immiscible fluids

The multiphase flow problem may be straightforward taken over by merely setting out the coefficients showed up in Eq. (37). In fact, one of the remarkable features of the approach introduced hitherto is the adaptability through clear-cut manipulation of the abovementioned equations.

3.1. Solid phase

The Eq. (16), when rephrased for multiphase flow, becomes:

$$(C_{ijkl}\dot{\epsilon}_{kl})_j + \sum_{K=1,N+1} a_K \dot{p}_i^K + \dot{F}_i = 0 \tag{38}$$

Where $a_K = \alpha S_K$ with $K = 1, \dots, N + 1$ for N immiscible aqueous and non aqueous fluids and one gaseous phase.

3.2. Fluid phase

The govern equation for N immiscible fluids, with $I = 1, \dots, N, J = 1, \dots, N, K = 1, \dots, N + 1$ and $L = N + 1$ is:

$$a_I \dot{\epsilon}_{ii} - \sum_{K=1,N+1} a_{IK} \dot{p}^K + \frac{1}{\rho^I} \left(\rho^I \frac{K_{ij}^I}{\mu^I} p_j^I \right)_{,i} = 0 \tag{39}$$

being

$$\begin{aligned} a_{II} &= S_I \left[\frac{n}{K_I} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_I + \sum_{J=1,N} \frac{C_{sJ}}{n} (dp^J - dp^I) \right) \right] + \frac{(\alpha - n)}{K_T} \\ a_{IJ} &= S_I S_J \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \right] \quad \forall I \neq J \\ a_{IL} &= S_I \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_L - \sum_{J=1,N} \frac{C_{sJ}}{n} (dp^J - dp^I) \right) \right] \end{aligned} \tag{40}$$

and

$$C_{sJ} = n \frac{dS_J}{dp^J} \tag{41}$$

3.3. Gaseous phase

Considering multi-fluids, the governing equation for the gaseous phase is:

$$a_L \dot{\epsilon}_{ii} - \sum_{K=1,N+1} a_{LK} \dot{p}^K + \frac{1}{\rho^L} (D_{ij}^* p_j^L)_{,i} = 0 \quad \text{with } L = N + 1 \tag{42}$$

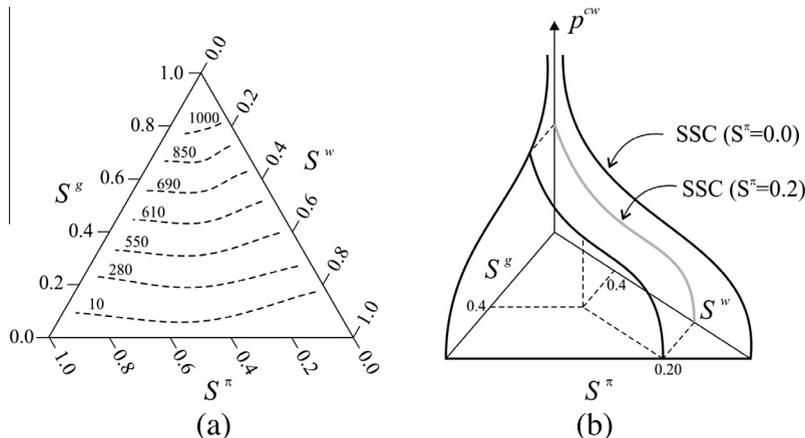


Fig. 6. (a) Map of estimated matrix suction using linear interpolation, (b) 3D graphics of the matrix suction.

with

$$a_{Lj} = \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_L \left(S_j + \frac{C_{sj}}{n} (dp^L - dp^I) \right) \right] \quad (43)$$

$$a_{LL} = \left[\frac{nS_L}{P} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_L \left(S_L - \sum_{j=1,N} \frac{C_{sj}}{n} (dp^L - dp^I) \right) + \frac{(\alpha - n)}{K_T} \right]$$

Considerations emphasized for C_{sw} and $C_{s\pi}$ are likewise valid for C_{sl} . However, provided that multiphase flow involves in advance an elevated number of fluids, clearly, evaluating these coefficients is too complex an experimental task.

4. Versatility

One turning point of any mathematical model is its capacity to withstand those situations in which the outcomes respond to a well-known pattern whether quantitative or qualitative. To validate Eqs. (38), (39) and (42), different fringe cases were suggested and the adaptability of the model is put forward under consideration.

4.1. Fully saturated case

Considering the saturated case, it entails setting $N = 1, S_w = 1, S_g = 0$. Attending these prerogatives, the relevant coefficients have the following values:

$$S_w = 1 \Rightarrow a_1 = \alpha \wedge a_2 = a_{12} = a_{21} = a_{22} = 0 \wedge a_{11} = \frac{nS_w}{K_w} - \frac{(\alpha - n)}{K_s} \quad (44)$$

and the full system for one fluid, i.e. Eq. (37), converge to the same model presented in Ref. [24]

4.2. Non saturated without pollutant phase

Considering the absence of a pollutant phase ($S_\pi = 0$ and $S_w + S_g = 1$), the full system size is straightforward reduced to the one presented in Ref. [25], with the vanishing of the following coefficients:

- $a_2 = a_{12} = a_{22} = a_{23} = a_{32} = 0$
- $a_1, a_3, a_{11}, a_{13}, a_{31}$ and a_{33} amount those presented in Ref. [25]

When the saturation change with suction is disregarded (a soil with high or low water saturation) the system is further reduced to a symmetric one with coefficients $a_{13} = a_{31}$ as well as a noteworthy simplification of a_{11} and a_{33} is endured, everything in accordance with Ref. [25]

5. A consistency check: two tantamount fluids with air presence

Consider the situation in which two immiscible phases with identical physical and chemical characteristics and an additional gaseous phase. The overall behavior of the system should equal the solution in Ref. [24] which, in turn, should lead to a reduction in the system of Eq. (37) according to Ref. [24]. In order to undertake this endeavor, the following quantities must be introduced: $S_w = \beta S_l, S_\pi = \gamma S_l, w = \pi$ and $\beta + \gamma = 1$.

Being S_l the total liquid fluid saturation, and being β and γ the proportional coefficients that add up to one. Due to the fact that both fluids were almost equivalent, Eq. (37) may be reformulated consequently.

5.1. Solid phase

With the assumption of tantamount liquid phases, being both phases water for example, there are no reason for supposing that p^π and p^w will bring forth different values throughout the consolidation process. Then, it would be mandatory to set $p^\pi = p^w$. Now, the first equation of the Eq. (37) conveys:

$$(C_{ijkl}\dot{\epsilon}_{kl})_j + (a_1 + a_2)\dot{p}_i^w + a_3\dot{p}_i^g + \dot{F}_i = 0 \quad (45)$$

with

$$a_1 + a_2 = \alpha S_w + \alpha S_\pi = \alpha(S_w + S_\pi) = \alpha(\beta S_l + \gamma S_l) = \alpha S_l \quad (46)$$

5.2. Water phase

Again, if are to be two waters, $p^\pi = p^w$ is introduced and:

$$a_1 \dot{\epsilon}_{ii} - (a_{11} + a_{12}) \dot{p}^w - a_{13} \dot{p}^g + \frac{1}{\rho^w} \left(\rho^w \frac{K_{ij}^w}{\mu^w} p_j^w \right)_i = 0 \quad (47)$$

where

$$a_1 = \alpha S_w = \alpha \beta S_l \quad (48)$$

$$a_{11} + a_{12} = S_w \left[\frac{n}{K_w} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_w + \frac{C_{sw}}{n} (dp^g - dp^w) + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) + \frac{(\alpha - n)}{K_T} \right] \\ + S_w S_\pi \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \right] \quad (49)$$

with

$$C_{sw} = n \frac{dS_w}{dp^{cw}} = n \frac{dS_l}{dp^{cw}} \frac{S_w}{S_l}; \quad C_{s\pi} = n \frac{dS_\pi}{dp^{c\pi}} = n \frac{dS_l}{dp^{c\pi}} \frac{S_\pi}{S_l} \quad (50)$$

In the previous, both β and γ were considered constant throughout the process. Putting Eq. (50) in Eq. (49) and after some algebraic work, it is obtained

$$a_{11} + a_{12} = S_w \left[\frac{n}{K_w} + \frac{(\alpha - n)}{K_s} \left(S_w + (dp^g - dp^w) \frac{dS_l}{dp^{cw}} + S_\pi \right) - \frac{(\alpha - n)}{K_T} \left((dp^g - dp^w) \frac{dS_l}{dp^{cw}} - S_g \right) \right] \quad (51)$$

5.3. Pollutant phase

With $p^\pi = p^w$, it is obtained

$$a_2 \dot{\epsilon}_{ii} - (a_{21} + a_{22}) \dot{p}^\pi - a_{23} \dot{p}^g + \frac{1}{\rho^\pi} \left(\rho^\pi \frac{K_{ij}^\pi}{\mu^\pi} p_j^\pi \right)_i = 0 \quad (52)$$

$$a_2 = \alpha S_\pi = \alpha \gamma S_l \quad (53)$$

$$a_{21} + a_{22} = S_\pi \left[\frac{n}{K_\pi} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_\pi + \frac{C_{s\pi}}{n} (dp^g - dp^w) + \frac{C_{sw}}{n} (dp^g - dp^\pi) \right) + \frac{(\alpha - n)}{K_T} \right] \\ + S_w S_\pi \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \right] \quad (54)$$

Again, regarding Eq. (50) and doing some algebraic duty, it is obtained

$$a_{21} + a_{22} = S_\pi \left[\frac{n}{K_w} + \frac{(\alpha - n)}{K_s} \left(S_w + (dp^g - dp^w) \frac{dS_l}{dp^{cw}} + S_\pi \right) - \frac{(\alpha - n)}{K_T} \left((dp^g - dp^w) \frac{dS_l}{dp^{cw}} - S_g \right) \right] \quad (55)$$

5.4. Gaseous phase

For this phase, all manipulation is constrained to add up coefficients a_{31} and a_{32} and to include Eq. (50) in the resulting addition as well as in a_{33} . Then

$$a_{31} + a_{32} = \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_g \left(S_l + (dp^g - dp^w) \frac{dS_l}{dp^{cw}} \right) \right] \quad (56)$$

$$a_{33} = \left[\frac{n S_g}{P} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_g \left(S_g - (dp^g - dp^w) \frac{dS_l}{dp^{cw}} \right) + \frac{(\alpha - n)}{K_T} \right] \quad (57)$$

5.5. The combination

Adding Eqs. (51) and (55), it is obtained:

$$a_{11} + a_{12} + a_{21} + a_{22} = S_l \left[\frac{n}{K_w} + \frac{(\alpha - n)}{K_s} \left(S_l + (dp^g - dp^w) \frac{dS_l}{dp^{cw}} \right) - \frac{(\alpha - n)}{K_T} \left((dp^g - dp^w) \frac{dS_l}{dp^{cw}} - S_g \right) \right] \tag{58}$$

Adding the second and third equations of Eq. (37) and substituting: $w = \pi = l$, $k_{\bar{r}} = k_{wi} + k_{\pi i}$ yields

$$(a_1 + a_2) \dot{\hat{e}}_{ii} - (a_{11} a_{12} + a_{21} + a_{22}) \dot{p}^l - (a_{13} + a_{23}) \dot{p}^g + \frac{1}{\rho^l} \left(\rho^l \frac{K_{ij}^l}{\mu^l} p_j^l \right)_i \tag{59}$$

Both Eqs. (58) and (59) are equivalent to a_{11} and to the second of Eq. (34) in Ref. [25] respectively. That is, starting from a four phase system of Eq. (37) and after introducing simply an additional relationship among two unknowns (i.e. $p^w = p^\pi$) from a concrete practical situation (both fluids might be indeed equal and no longer immiscible), the model boils down to the one addressed in the aforementioned reference which stood for a three phase scheme with only one fluid phase. This is too important a conclusion, all the more because the convergence of the four phase system to the simpler case of Ref. [25] when only one fluid is present, allows validating not only the equations regarding relative permeability along with the ensuing mathematic but also all those stress states that involve the pollutant pore pressure, putting forward the consistency of the present extension

6. Finite element implementation

Applying the Galerkin method to the system of Eq. (37) and using the finite element technique, it is obtained the following system of ordinary differential equations:

$$\begin{aligned} \mathbf{K} \dot{\mathbf{u}} + \mathbf{C}_{sw} \dot{\mathbf{p}}^w + \mathbf{C}_{sg} \dot{\mathbf{p}}^g + \mathbf{C}_{s\pi} \dot{\mathbf{p}}^\pi &= \dot{\mathbf{F}}_s \\ \mathbf{C}_{ws} \dot{\mathbf{u}} + \mathbf{P}_{ww} \dot{\mathbf{p}}^w + \mathbf{C}_{wg} \dot{\mathbf{p}}^g + \mathbf{C}_{w\pi} \dot{\mathbf{p}}^\pi + \mathbf{H}_{ww} \bar{\mathbf{p}}^w &= \dot{\mathbf{F}}_w \\ \mathbf{C}_{gs} \dot{\mathbf{u}} + \mathbf{C}_{wg} \dot{\mathbf{p}}^w + \mathbf{P}_{gg} \dot{\mathbf{p}}^g + \mathbf{C}_{g\pi} \dot{\mathbf{p}}^\pi + \mathbf{H}_{gg} \bar{\mathbf{p}}^g &= \dot{\mathbf{F}}_g \\ \mathbf{C}_{\pi s} \dot{\mathbf{u}} + \mathbf{C}_{\pi w} \dot{\mathbf{p}}^w + \mathbf{C}_{\pi g} \dot{\mathbf{p}}^g + \mathbf{P}_{\pi\pi} \dot{\mathbf{p}}^\pi + \mathbf{H}_{\pi\pi} \bar{\mathbf{p}}^\pi &= \dot{\mathbf{F}}_\pi \end{aligned} \tag{60}$$

All the matrices involved in Eq. (60) are summarized in B.

7. Numerical example

In order to validate the numerical model attained in the previous paragraph as well as to give a numerical foothold to the discussion addressed in Section 5, the same one dimensional consolidation problem selected in Lewis et al. [38] was solved via the finite element based open code FECCUND (developed by the authors in Ref. [15,19,25,39]) using isoparametric elements with eight nodes for displacements and four for pore pressures.

The original soil column was assumed to be unsaturated with initial water saturation S_w equal to 0.52. The initial pore water pressure p^w was taken equal to 0.280 kPa and the boundary pore water pressure was instantaneously changed to 420 kPa at the surface. The column height was taken equal to 1 m. The material parameters were: Young's modulus $E = 173000$ kPa, Poisson's ratio $\nu = 0.4$, Permeability $k = 0.11456$ m/day, Void ratio $e = 0.4$, and the parameters for the saturation suction relationship (according to [32]) are: $a = 427$; $n = 0.794$; $m = 0.613$; $p_c^s = 3000$ kPa and $S_w^0 = 1.0$.

In Fig. 7a the geometry, the finite element mesh, the load and the boundary conditions were shown as well as the points where displacements and water saturation evolution were analyzed. The following value for the boundary conditions are: (1) Lateral surface: $u = 0.0$; (2) Bottom surface: $v = 0.0$; (3) Top surface: $p^w = 420$ kPa, $p^g =$ atmospheric pressure.

Now, the inclusion of the pollutant phase was carried in two ways: setting the pollutant saturation equal to 22% and water saturation equal to 30% (both summing up to 52%, the very value of the original column) and adopting different saturation suction curves for each other. Fig. 7(b) shows the vertical displacement vs. time for some selected points. Two important conclusion may be drawn here: (1) The present model in the absence of a pollutant phase ($S_\pi = 0$) bring forth a plot that properly adjusts to Lewis et al. [38]; (2) When both liquids are simulated but the same saturation suction curve is regarded, the situation is tantamount to *two waters* (or *two pollutants*) and the displacement curve fits exactly the one simulated in the previous point and thereby validating the convergence condition outlined in Section 5.

Fig. 8 shows how air, water and pollutant pore pressure change with time at a depth of 0.5 m within the same column. For the sake of simplicity the only difference in the genus of the saturation–suction curve for water phase and pollutant phase is the value of parameter m . For the water phase, m is maintained at 0.613 whereas for pollutant it is varied between 0.213 and 1.013. A remarkable fact is that, when m is the same for pollutant and water, both pore pressure curves merge into one which, in fact, matches the outcomes when $S_\pi = 0$ and $S_w = 52\%$. Another noteworthy feature is the meager variation of water pore pressure when the pollutant characteristic curve is altered.

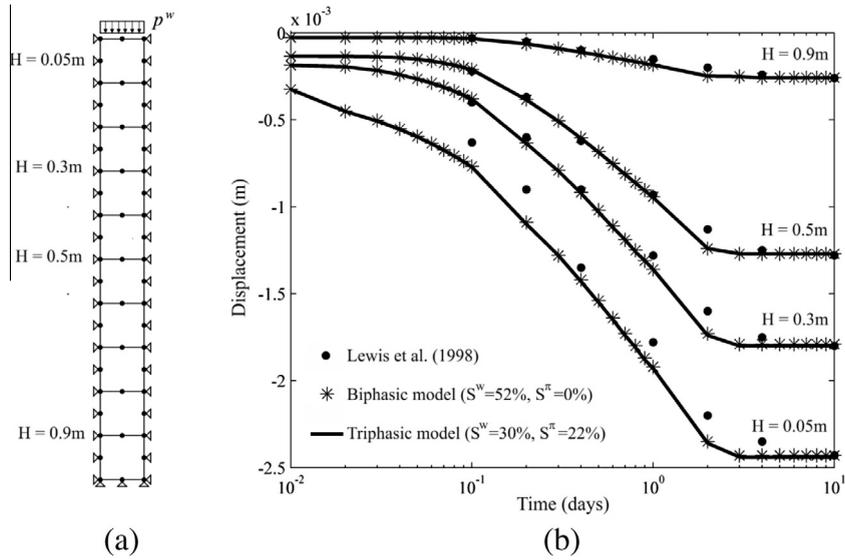


Fig. 7. Soil column: (a) geometry and boundary conditions; (b) displacement vs. time for selected points.

8. Final remarks

A general formulation and its finite element implementation and numerical solution for non-saturated soils consolidation with the presence of a pollutant phase were presented as well as a tentative extension to multi phase flow. The non-linear saturation suction and permeability-suction functions for both water and pollutant phase were incorporated along with the Galerkin-based finite element model. The governing equation, in terms of displacement and fluid pressures, result in a coupled non-linear partial differential equation system. Some of the coupling factors, i.e., those due to the rate of suction-saturation relationship, can be straightforward decoupled by merely setting certain coefficients to special values when facing conditions of high liquid saturation which in turn, leads to simple symmetric formulations as indicated by Di Rado et al. [25]. Widely,

1. The stress state-base combination model can be straightforward extended to encompass a wider set of diverse fluids heading always to a flexible mathematical framework in the sense of its coupling-decoupling capability with the ensuing waning of computational requirements.
2. Simple expressions for the compressibility of solid structure with respect to a change in water suction along with the compressibility of solid structure with respect to a change in pollutant suction were introduced.
3. For evaluating the coefficients relating fluid saturation rates with respect to suction, i.e. C_{sw} and C_{sp} , (Eq. (33)), a revision of the classical laboratory test was introduced. Furthermore using the upshots thereby drawn and adding a suitable interpolation methods, a handy ternary map for both C_{sw} and C_{sp} , may be outlined.

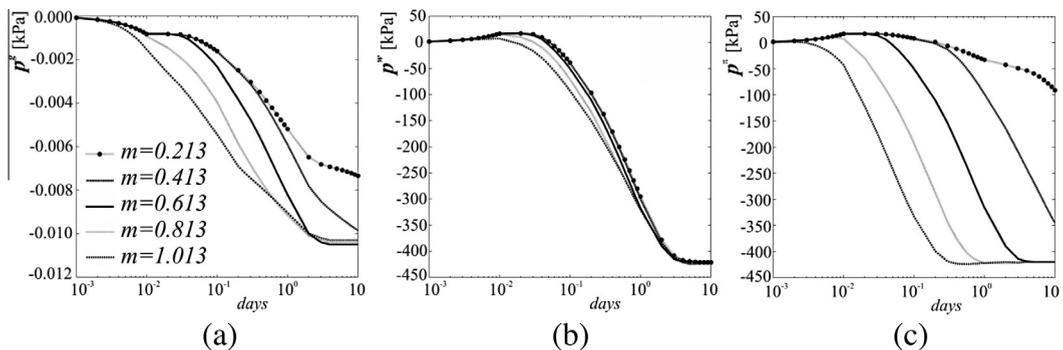


Fig. 8. Pore pressure at 0.5 m of depth for: (a) gaseous phase; (b) water phase; (c) pollutant phase.

4. An extension of the mathematical approach to “N” immiscible pollutant phases was outlined. However, some forewarnings towards C_{si} evaluation were posed.
5. In view of the herein introduced framework encompass that given by [25], all the noteworthy features depicted for that model naturally hold for the present case.
6. The mathematical framework validation shows an appealing behavior and wide adaptability when fringe selected situation are induced.

Appendix A. Coefficients involved in Eq. (37)

$$\begin{aligned}
 a_1 &= \alpha S_w, \quad a_2 = \alpha S_\pi, \quad a_3 = \alpha S_g \\
 a_{11} &= S_w \left[\frac{n}{K_w} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_w + \frac{C_{sw}}{n} (dp^g - dp^w) + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) + \frac{(\alpha - n)}{K_T} \right] \\
 a_{12} &= S_w S_\pi \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \right] \\
 a_{13} &= S_w \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_g - \frac{C_{sw}}{n} (dp^g - dp^w) - \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) \right] \\
 a_{21} &= S_\pi S_w \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \right] \\
 a_{22} &= S_\pi \left[\frac{n}{K_\pi} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_\pi + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) + \frac{C_{sw}}{n} (dp^g - dp^w) \right) + \frac{(\alpha - n)}{K_T} \right] \\
 a_{23} &= S_\pi \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) \left(S_g - \frac{C_{s\pi}}{n} (dp^g - dp^\pi) - \frac{C_{sw}}{n} (dp^g - dp^w) \right) \right] \\
 a_{31} &= \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_g \left(S_w + \frac{C_{sw}}{n} (dp^g - dp^w) \right) \right] \\
 a_{32} &= \left[(\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_g \left(S_\pi + \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) \right] \\
 a_{33} &= \left[\frac{n S_g}{P} + (\alpha - n) \left(\frac{1}{K_s} - \frac{1}{K_T} \right) S_g \left(S_g - \frac{C_{sw}}{n} (dp^g - dp^w) - \frac{C_{s\pi}}{n} (dp^g - dp^\pi) \right) + \frac{(\alpha - n)}{K_T} \right]
 \end{aligned}$$

Appendix B. Finite element matrix in Eq. (60)

$$\dot{u}_i = \mathbf{N}^u \dot{\mathbf{u}} \quad \dot{p}^g = \mathbf{N}^p \dot{\mathbf{p}}^g \quad \dot{p}^w = \mathbf{N}^p \dot{\mathbf{p}}^w \quad \dot{p}^\pi = \mathbf{N}^p \dot{\mathbf{p}}^\pi \quad \mathbf{m} = \{1, 1, 1, 0, 0, 0\}$$

$$\begin{aligned}
 \mathbf{K} &= \int_\Omega \mathbf{B}^{uT} \mathbf{D} \mathbf{B}^u d\Omega & \mathbf{P}_{ww} &= - \int_\Omega a_{11} \mathbf{N}^{pT} \mathbf{N}^p d\Omega \\
 \mathbf{P}_{\pi\pi} &= - \int_\Omega a_{22} \mathbf{N}^{pT} \mathbf{N}^p d\Omega & \mathbf{P}_{gg} &= - \int_\Omega a_{33} \mathbf{N}^{pT} \mathbf{N}^p d\Omega \\
 \mathbf{C}_{sw} &= \int_\Omega a_1 \mathbf{B}^{uT} \mathbf{m} \mathbf{N}^p d\Omega & \mathbf{C}_{ws} &= \int_\Omega a_1 \mathbf{N}^{pT} \mathbf{m} \mathbf{B}^u d\Omega \\
 \mathbf{C}_{s\pi} &= \int_\Omega a_2 \mathbf{B}^{uT} \mathbf{m} \mathbf{N}^p d\Omega & \mathbf{C}_{\pi s} &= \int_\Omega a_2 \mathbf{N}^{pT} \mathbf{m} \mathbf{B}^u d\Omega \\
 \mathbf{C}_{sg} &= \int_\Omega a_3 \mathbf{B}^{uT} \mathbf{m} \mathbf{N}^p d\Omega & \mathbf{C}_{gs} &= \int_\Omega a_3 \mathbf{N}^{pT} \mathbf{m} \mathbf{B}^u d\Omega \\
 \mathbf{C}_{\pi w} &= - \int_\Omega a_{21} \mathbf{N}^{pT} \mathbf{N}^p d\Omega & \mathbf{C}_{w\pi} &= - \int_\Omega a_{12} \mathbf{N}^{pT} \mathbf{N}^p d\Omega \\
 \mathbf{C}_{gw} &= - \int_\Omega a_{31} \mathbf{N}^{pT} \mathbf{N}^p d\Omega & \mathbf{C}_{wg} &= - \int_\Omega a_{13} \mathbf{N}^{pT} \mathbf{N}^p d\Omega \\
 \mathbf{C}_{g\pi} &= - \int_\Omega a_{32} \mathbf{N}^{pT} \mathbf{N}^p d\Omega & \mathbf{C}_{\pi g} &= - \int_\Omega a_{23} \mathbf{N}^{pT} \mathbf{N}^p d\Omega \\
 \dot{\mathbf{F}}_w &= - \int_{\Gamma_w} \mathbf{N}^{pT} \dot{\mathbf{q}}_w d\Gamma & \mathbf{H}_{ww} &= \int_\Omega \nabla \mathbf{N}^{pT} \frac{\mathbf{K}^w}{\mu^w} \nabla \mathbf{N}^p d\Omega \\
 \dot{\mathbf{F}}_\pi &= - \int_{\Gamma_\pi} \mathbf{N}^{pT} \dot{\mathbf{q}}_\pi d\Gamma & \mathbf{H}_{\pi\pi} &= \int_\Omega \nabla \mathbf{N}^{pT} \frac{\mathbf{K}^\pi}{\mu^\pi} \nabla \mathbf{N}^p d\Omega \\
 \dot{\mathbf{F}}_g &= - \int_{\Gamma_g} \mathbf{N}^{pT} \dot{\mathbf{q}}_g d\Gamma & \mathbf{H}_{gg} &= \int_\Omega \nabla \mathbf{N}^{pT} \frac{\mathbf{D}^*}{P} (1 - S_w) n \nabla \mathbf{N}^p d\Omega \\
 \dot{\mathbf{F}}_s &= \int_\Omega \mathbf{N}^{uT} \dot{\mathbf{b}} d\Omega + \int_{\Gamma_g} \mathbf{N}^{uT} \dot{\mathbf{t}} d\Omega
 \end{aligned}$$

where:

\mathbf{N}^u	:	Vector containing the interpolation functions of the displacements components.
\mathbf{N}^p	:	Vector containing the interpolation functions of the water and gas pressures.
\mathbf{B}^u	:	Matrix relating strain and displacement components.
\mathbf{D}	:	Constitutive matrix.
\mathbf{b}	:	Body force vector acting on the element domain.
\mathbf{t}	:	Surface force vector acting on the element boundary.
$\nabla \mathbf{N}$:	Gradient of the interpolation functions.
\mathbf{q}_w	:	Water flux through the boundary element.
\mathbf{q}_π	:	Pollutant flux through the boundary element.
\mathbf{q}_g	:	Air flux through the boundary element.

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