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The k-Fractional Logistic Equation with k-Caputo Derivative

Rubén A. Cerutti

Faculty of Exact Sciences
National University of the Northeast
Av. Libertad 5540 (3400); Corrientes, Argentina

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Abstract

A generalization of the fractional logistic equation by using k-Caputo derivative is introduced. Also a solution that can be expressed en terms of the k-Mittag-Leffler function is obtained.
The development of this paper has been done on the basis what has been done by Camargo and Bruno-Alfonso in [3].

Keywords: k-Fractional calculus, k-Mittag-Leffler function, k-Riemann-Liouville fractional derivative

I Preliminaries

Generalizations of the classical Gamma function and the Pochhammer's symbol were introduced by Diaz and Pariguan [1].Since then has appeared a significant number of papers extended definitions of fractional order of integrodifferential operators using these generalizations given rise to what may be called the k-Fractional calculus.
Let us remember the definition of the k-Gamma function introduced in [1] and its relations with the classical Gamma function.

Definition 1 *Let z be a complex number that* $\mathrm{Re}(z) > 0$*. The k-Gamma function is given by the following integral*

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt. \tag{I.1}$$

The relationship between $\Gamma_k(z)$ and the classical $\Gamma(z)$ is expressed by

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{I.2}$$

It can be seen that $\Gamma_k(z)$ is such that $\Gamma_k(z) \to \Gamma(k)$ as $k \to 1$.
The function $\Gamma_k(z)$ satisfies the following properties

1. $\Gamma_k(z+k) = z\Gamma_k(z)$,
2. $\Gamma_k(k) = 1$.

For further development of our article, we need to introduce the k-Pochhammer symbol $(\gamma)_{n,k}$ due to Diaz and Pariguan [1] given by the following

Definition 2 *Let γ be a complex number, $k > 0$ and $n \in \mathbb{N}$. The k-Pochhammer symbol $(\gamma)_{n,k}$ is*

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)...(\gamma+(n-1)k). \tag{I.3}$$

It can be seen that $(\gamma)_{n,1} = (\gamma)_n$, the classical Pochhammer symbol.
Also verifies the following properties

1. $(\gamma)_{n,k} = k^n \left(\frac{\gamma}{k}\right)_n,$ (I.4)

2. $(\gamma)_{n,k} = \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}.$ (I.5)

According [2], now we introduce the k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$ by the following

Definition 3 *Let $k \in \mathbb{R}^+$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.*
The k-Mittag-Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$ is given by the following series

$$E^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\ z^n}{\Gamma_k(\alpha n+\beta)n!} \tag{I.6}$$

where $(\gamma)_{n,k}$ is the k-Pochhammer symbol given in (I.3) and $\Gamma_k(z)$ is the k-gamma function given in (I.1).

When we consider $k = \gamma = 1$, (I.6) reduce to the classical two parameters Mittag-Leffler function

$$E^1_{1,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}(z) \tag{I.7}$$

and, if $\alpha = \beta = \gamma = k = 1$, it result the exponential function

$$E^1_{1,1,1}(z) = e^z$$

The Laplace transform of a function closely related to the k-Mittag-Leffler function is know. It is the following (cf [2])

$$\mathcal{L}\left\{ z^{\frac{\beta}{k}-1} E^{\gamma}_{k,\alpha,\beta}\left(k^{\frac{\alpha}{k}-1} az\right)\right\}(s) = \frac{s^{\frac{\alpha}{k}\frac{\gamma}{k}} k^{1-\frac{\beta}{k}}}{s^{\frac{\beta}{k}}\left(s^{\frac{\alpha}{k}} - a\right)^{\frac{\gamma}{k}}}; \tag{I.8}$$

for $\left|as^{-\frac{\alpha}{k}}\right| < 1$.

Analogously to what was done with the Riemann-Liouville fractional derivative, we now introduce a modification of the Caputo derivative, modification that involves the k-Riemann-Liouville singular kernel.
Then, we put, by

Definition 4 *Let f be a function at least n times differentiable, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$.*
The k-Caputo fractional derivative of order α of the function f is given by

$${}^cD^{\alpha}_k f(t) = I^{n-\alpha}_k f^{(n)}(t), \tag{I.9}$$

where $f^{(n)}(t) = D^n f(t)$ is the ordinary derivative, and $I^{n-\alpha}_k$ is the k-Riemann-Liouville integral.

Remembering the definition of the k-Riemann-Liouville singular kernel

$$J_{\gamma,k}(t) = \begin{cases} \frac{t^{\frac{\gamma}{k}-1}}{k\Gamma_k(\gamma)} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{I.10}$$

the formula (I.9) may be expressed as a convolution

$${}^cD^{\alpha}_k f(t) = J_{n-\alpha,k} * f^{(n)}(t). \tag{I.11}$$

The Laplace transform of $J_{\gamma,k}(z)$ is (cf [6])

$$\mathcal{L}\left[\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(z)}\right](s) = (ks)^{-\frac{\alpha}{k}}. \tag{I.12}$$

II The k-fractional logistic equation

After the considerations made by Camargo and Bruno-Alfonso in [3], we present the equation

$$ {}^{c}D_k^{\alpha} f(t) = k^{\frac{\alpha-1}{k}} \lambda[1 - f(t)] \tag{II.1}$$

Where ${}^{c}D_k^{\alpha}$ denote the k-fractional derivative introduced in (I.9), $0 < \alpha < 1$. Equivalently, we have

$$ {}^{c}I_k^{\frac{1-\alpha}{k}} f'(t) = k^{\frac{\alpha-1}{k}} \lambda(1 - f(t)) \tag{II.2}$$

By applying the Laplace transform to the equation (II.2) and taking into account some of their basic properties and the expression (I.12), from the left hand side, we have

$$\begin{aligned} \mathcal{L}[J_{1-\alpha,k}] \cdot \mathcal{L}(f') &= (ks)^{-\frac{(1-\alpha)}{k}} \cdot \mathcal{L}(f') = \\ &= (ks)^{\frac{\alpha-1}{k}} \{sF(s) - f(0)\} \end{aligned} \tag{II.3}$$

where $F(s) = \mathcal{L}(f)$

From the right hand member of (II.1) we have

$$ k^{\frac{\alpha-1}{k}} \mathcal{L}[\lambda(1 - f(t))] = k^{\frac{\alpha-1}{k}} \lambda \left(\frac{1}{s} - F(s) \right) \tag{II.4}$$

Thus, from (II.3) and (II.4) it results

$$ s^{\frac{\alpha-1}{k}} \{sF(s) - f(0)\} = \lambda \left\{ \frac{1}{s} - F(s) \right\}, $$

$$ s^{\frac{\alpha-1+k}{k}} \cdot F(s) - s^{\frac{\alpha-1}{k}} f(0) = \lambda \left\{ \frac{1}{s} - F(s) \right\} $$

$$ \left\{ s^{\frac{\alpha-1+k}{k}} + \lambda \right\} F(s) = s^{\frac{\alpha-1}{k}} f(0) + \frac{\lambda}{s} $$

$$ F(s) = \frac{s^{\frac{\alpha-1}{k}}}{\left(s^{\frac{\alpha-1+k}{k}} + \lambda \right)} f(0) + \frac{\lambda s^{-1}}{\left(s^{\frac{\alpha-1+k}{k}} + \lambda \right)} \tag{II.5}$$

Note that if in (I.8), $\gamma = k$ is considered, we have

$$ \mathcal{L} \left\{ z^{\frac{\beta}{k}-1} E_{k,\alpha,\beta}^{k} \left(k^{\frac{\alpha}{k}-1} a z^{\alpha} \right) \right\} = \frac{s^{\frac{\alpha}{k}} k^{1-\frac{\beta}{k}}}{s^{\frac{\beta}{k}} \left(s^{\frac{\alpha}{k}} - a \right)} \tag{II.6}$$

and, taking into account that $(k)_{n,k} = k^n n!$, the left hand member of (II.6) is

$$\mathcal{L}\left\{ z^{\frac{\beta}{k}-1} \sum_{n=0}^{\infty} \frac{k^n \left(k^{\frac{\alpha}{k}-1} a z^{\alpha}\right)^n}{\Gamma_k(\alpha n+\beta)} \right\} = \mathcal{L}\left\{ z^{\frac{\beta}{k}-1} E_{k,\alpha,\beta}\left(k^{\frac{\alpha}{k}} a z^{\alpha}\right) \right\} \tag{II.7}$$

Let $\frac{s^{\frac{\alpha-1}{k}}}{\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)}$; and consider

$$\frac{s^{\frac{\alpha-1}{k}+1-1}}{\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \frac{s^{\frac{\alpha-1+k}{k}} s^{-1}}{\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \frac{s^{\frac{\alpha-1+k}{k}}}{s\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \mathcal{L}\left[E_{k,\frac{\alpha-1+k}{k}}\left(-k^{\frac{\alpha-1+k}{k^2}-1} \lambda t^{\frac{\alpha-1+k}{k}}\right)\right] \tag{II.8}$$

And

$$\frac{s^{-1}}{\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \frac{s^{\frac{\alpha-1+k}{k}+\left(-1-\frac{\alpha-1+k}{k}\right)}}{\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \frac{s^{\frac{\alpha-1+k}{k}}}{s^{1+\frac{\alpha-1+k}{k}}\left(s^{\frac{\alpha-1+k}{k}}+\lambda\right)} = \mathcal{L}\left\{ t^{1+\frac{\alpha-1+k}{k}-1} E_{k,\frac{\alpha-1+k}{k},1+\frac{\alpha-1+k}{k}}\left[-k^{\frac{\alpha-1+k}{k^2}-1} \lambda t^{\frac{\alpha-1+k}{k}}\right] \right\} \tag{II.9}$$

From (II.5), (II.8) and (II.9) we have

$$F(s) = f(0)\mathcal{L}\left(t^{\frac{\alpha-1+k}{k}} E_{k,\frac{\alpha-1+k}{k},1+\frac{\alpha-1+k}{k}}\left[-k^{\frac{\alpha-1+k}{k^2}-1} \lambda t^{\frac{\alpha-1+k}{k}}\right]\right) + +\lambda\mathcal{L}\left(E_{k,\frac{\alpha-1+k}{k}}\left[-k^{\frac{\alpha-1+k}{k^2}} \lambda t^{\frac{\alpha-1+k}{k}}\right]\right) \tag{II.10}$$

and, by applying the inverse Laplace transform it results

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = f(0) \cdot E_{k,\frac{\alpha-1+k}{k}}\left[-k^{\frac{\alpha-1+k}{k^2}} \lambda t^{\frac{\alpha-1+k}{k}}\right] + +\lambda t^{\frac{\alpha-1+k}{k}} E_{k,\frac{\alpha-1+k}{k},1+\frac{\alpha-1+k}{k}}\left[-k^{\frac{\alpha-1+k}{k^2}} \lambda t^{\frac{\alpha-1+k}{k}}\right] \tag{II.11}$$

It can be seen that, when $k = 1$, from (II.11), we have

$$f(t) = f(0)E_{\alpha}\left(-\lambda t^{\alpha}\right) + \lambda t^{\alpha} E_{\alpha,\alpha+1}\left(-\lambda t^{\alpha}\right) \tag{II.12}$$

which is the result given in [3].
We can see that
$E_{k,\frac{\alpha-1+k}{k},\frac{\alpha-1+k}{k}+1}\left(-(kt)^{\frac{\alpha-1+k}{k}}\lambda\right)$ it can be written as

$$E_{k,\frac{\alpha-1+k}{k},\frac{\alpha-1+k}{k}+1}\left(-(kt)^{\frac{\alpha-1+k}{k}}\lambda\right)=$$

$$=-\frac{1}{\lambda(kt)^{\frac{\alpha-1+k}{k}}}\sum_{n=0}^{\infty}\frac{\left(-(kt)^{\frac{\alpha-1+k}{k}}\lambda\right)^{n+1}}{\Gamma_k\left((n+1)\left(\frac{\alpha-1+k}{k}\right)+1\right)}=$$

$$=-\frac{1}{\lambda(kt)^{\frac{\alpha-1+k}{k}}}\left\{-\frac{1}{\Gamma_k(1)}+\sum_{n=0}^{\infty}\frac{\left(-(kt)^{\frac{\alpha-1+k}{k}}\lambda\right)^{n}}{\Gamma_k\left((n+1)\frac{\alpha-1+k}{k}+1\right)}\right\}=$$

$$=-\frac{1}{\lambda(kt)^{\frac{\alpha-1+k}{k}}}\left\{-\frac{1}{\Gamma_k(1)}+E_{k,\frac{\alpha-1+k}{k}}\left(-\lambda(kt)^{\frac{\alpha-1+k}{k}}\right)\right\}. \qquad \text{(II.13)}$$

Then, replacing (II.13) in (II.11) it results

$$f(t)=f(0)E_{k,\frac{\alpha-1+k}{k}}\left(-\lambda(kt)^{\frac{\alpha-1+k}{k}}\right)+\frac{1}{k^{\frac{\alpha-1+k}{k}}}\left\{-\frac{1}{\Gamma_k(1)}+E_{k,\frac{\alpha-1+k}{k}}\left(-\lambda(kt)^{\frac{\alpha-1+k}{k}}\right)\right\}$$

or equivalently

$$f(t)=E_{k,\frac{\alpha-1+k}{k}}\left(-\lambda(kt)^{\frac{\alpha-1+k}{k}}\right)\left\{f(0)-\frac{1}{k^{\frac{\alpha-1+k}{k}}}\right\}+\frac{1}{k^{\frac{\alpha-1+k}{k}}}\cdot\frac{1}{\Gamma_k(1)} \qquad \text{(II.14)}$$

Taking $k=1$, it results

$$f(t)=E_{\alpha}\left(\lambda(-t)^{\alpha}\right)\{f(0)-1\}+1 \qquad \text{(II.15)}$$

when $\lambda=1$, coincides with the results due to Figueiredo Camargo and Bruno-Alfonso [3]

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