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A GENERALIZATION OF THE KINETIC EQUATION USING THE PRABHAKAR-TYPE OPERATORS

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Abstract. Fractional kinetic equations are investigated in order to describe the various phenomena governed by anomalous reaction in dynamical systems with chaotic motion. Many authors have provided solutions of various families of fractional kinetic equations involving special functions. Here, in this paper, we aim at presenting solutions of certain general families of fractional kinetic equations using Prabhakar-type operators. The idea of present paper is motivated by Tomovski et al. [21].

1. Introduction

The Fractional Calculus (FC) can be considered as a generalization of the classical differential calculus, in which one basically studies operators of differentiation and integration of non-integer orders. In the last decades, the FC stopped being only a field of theoretical study to happen to be a powerful tool with which scientists study and model phenomena of the real life. Several applications of Fractional Calculus in different branches of science arise day to day. A very interesting work that brings together many applications in both science and engineering is that of Machado et. al. [35].

From the applications, questions arise naturally related to the modeling through differential equations of non-integer orders. An interesting question in proposing such equations is that it allows interpolating, for example, between the diffusion equation and the wave equation and studying intermediate phenomena. Another question is that the use

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of derivatives of non-integer orders allows us to raise questions about nonlocal phenomena in space and phenomena with memory over time.

On the other hand, the fractional derivatives appear naturally in the description of different physical phenomena. For example, the time fractional derivatives are equivalent to infinitesimal generators of generalized time fractional evolutions, which arise in the transition from microscopic to macroscopic time scales (cf. [23, 24, 34]).

2. Preliminaries

We begin by tracing the fractional kinetic equation. Consider an arbitrary reaction characterized by a time dependent quantity $N = N(t)$. It is possible to calculate rate of change dN/dt to be a balance between the destruction rate d and the production rate p of N, that is, $dN/dt = -d + p$. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself: $d = d(N)$ and $p = p(N)$. This dependence is complicated since the destruction or the production at a time t depends on not only $N(t)$ but also the past history $N(\eta)$, $\eta < t$, of the variable N. This may be formally represented by the following equation (see [22]):

$$\frac{dN}{dt} = -d\left(N_t\right) + p\left(N_t\right), \tag{1}$$

where N_t denotes the function defined by $N_t\left(t^*\right) = N\left(t - t^*\right),\ t^* > 0$. Haubold and Mathai [22] studied a special case of the equation (1) as in the following form:

$$\frac{dN_i}{dt} = -c_i\, N_i\left(t\right) \tag{2}$$

with the initial condition that $N_i\left(t = 0\right) = N_0$ is the number density of species i at time $t = 0$ and constant $c_i > 0$, which is known as standard kinetic equation. The solution of the equation (2) is easily seen to be given by

$$N_i\left(t\right) = N_0\, e^{-c_i t}. \tag{3}$$

Integration gives an alternative form of the equation (2) as follows:

$$N\left(t\right) - N_0 = c\, {}_0D_t^{-1} N\left(t\right), \tag{4}$$

where ${}_0D_t^{-1}$ is the standard integral operator.

The fractional generalization of the equation (4) is given as in the following form [22]:

$$N(t) - N_0 = c^{\nu}\, {}_0D_t^{-\nu} N(t)\,, \tag{5}$$

where ${}_0D_t^{-\nu}$ is the special case of the Riemann-Liouville fractional integral operator defined as

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s)\; ds, \quad (t>0, \Re(\nu)>0)\,. \tag{6}$$

Saxena and Kalla [27] considered the following fractional kinetic equation:

$$N(t) - N_0\, f(t) \;=\; -c^{v} \left({}_0D_t^{-v}\, N\right)(t) \quad (Re\,(v)\;>\;0)\,, \tag{7}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t=0$, c is a constant and $f \in\; L(0,\infty)$. By applying the Laplace transform to (7), we have

$$\mathcal{L}\left[N(t)\right](p) = N_0\, \frac{F(p)}{1+c^{v}\, p^{-v}} \;=\; N_0 \left(\sum_{n=0}^{\infty} (-c^{v})^{n}\; p^{-nv}\right) F(p) \;\left(n \in N_0,\; \left|\frac{c}{p}\right| <\; 1\right). \tag{8}$$

Tomovski et al. [21] provided the corrected version of the obviously erroneous solution of the fractional kinetic equation (7) given by Saxena and Kalla [[27], p. 508, Eqn. (3.2)] as follows:

$$N(t) \;=\; N_0 \left(f(t) + \sum_{n=1}^{\infty} \frac{(-c^{v})^{n}}{\Gamma(n\,v)} \left(t^{nv-1} \;*\; f(t)\right)\right), \tag{9}$$

or

$$N(t) \;=\; N_0 \left(f(t) + \sum_{n=1}^{\infty} (-c^{v})^{n} \left({}_0D_t^{-nv}\, f\right)(t)\right), \tag{10}$$

where the relationship between the Laplace convolution and the Riemann-Liouville fractional integral operator $\left({}_0D_t^{-v}\, f\right)$ with $a=0$, given as following :

$$t^{nv-1} * f(t) \;=\; \int_0^t (t-p)^{nv-1}\; f(p)\; dp \;=\; \Gamma(nv) \left({}_0D_t^{-nv}\, f\right)(t)\,, \quad (n \in\; N,\; Re\,(v)>0)\,. \tag{11}$$

The solution (10) provides the new version of the equation (7) by applying a technique which was employed earlier by Al. Saqabi and Tuan

[?] for solving fractional differintegral equations.
The general fractional kinetic differintegral equation given as following:

$$a\left(D_{0+}^{\alpha,\beta}N\right)(t)-N_0\,f(t)\,=\,b\left({}_0D_t^{-v}N\right)(t)\,, \tag{12}$$

under the initial condition

$$\left({}_0D_t^{-[(1-\beta)\,(1-\alpha)]}f\right)(0+)\,=\,c\,, \tag{13}$$

where a, b and c are constant and $f\in\,L\,(0,\infty)$.

In this paper, we consider a generalization of the following integral operator due to Prabhakar:

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a^+}\varphi)(x)=\int_a^x(x-t)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(x-t)^{\rho}]\varphi(t)dt\quad(x>a). \tag{14}$$

where $E^{\gamma}_{\rho,\mu}[\omega(x-t)^{\rho}]$, the Mittag-Leffler function defined in [3], will be replaced by the k-Mittag-Leffler function defined in [9], and given by

$$E^{\gamma}_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)}\frac{z^n}{n!} \tag{15}$$

where $k\in\mathbb{R}^+$; $\alpha,\beta,\gamma\in\mathbb{C}$; $Re(\alpha)>0, Re(\beta)>0$; $\Gamma_k(x)$ it is the k-Gamma function given by (20) and $(\gamma)_{n,k}=\frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}$ it is the Pochhammer k-symbol.

In the following, we will highlight some points needed for the sequel.

Definition 2.1. (Laplace Transform) *Let $f:\mathbb{R}^+\to\mathbb{R}$ an exponential order and piecewise continuous function, then the Laplace transform of f is*

$$\mathcal{L}\{f(t)\}(s):=\int_0^{\infty}e^{-st}f(t)dt. \tag{16}$$

The integral exist for $Re(s)>0$.

Definition 2.2. *Let $f\in L^1_{loc}[a,b]$ where $-\infty\leq a<t<b\leq\infty$. The Riemann-Liouville integral of order ν is defined as*

$$I^{\nu}f(t):=\frac{1}{\Gamma(\nu)}\int_a^t(t-\tau)^{\nu-1}f(\tau)d\tau\qquad\nu>0. \tag{17}$$

Definition 2.3. *Let* $f \in L^1[a,b]$, $-\infty \le a < t < b \le \infty$ *and* $I^{n-\nu} f(t) \in W^{n,1}[a,b]$, $n = [\nu]+1$, $\nu > 0$

The Riemann-Liouville derivative of order ν, *is given by*

$$D^{\nu} f(t) := \left(\frac{d}{dt}\right)^n I^{n-\nu} f(t), \tag{18}$$

where $W^{n,1}[a,b] = \{f \in L^1[a,b] : f^{(n)} \in L^1[a,b]\}$ *is the Sobolev space.*

In 2012 Mubeen and Habbibulah (cf.[17]) introduced the k-Riemann-Liouville fractional integral given by

Definition 2.4. *Let* $\alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$ *such that* $n-1 < \alpha < n$, $f \in L^1([0,\infty))$. *Then the k-Riemann-Liouville fractional integral of f is*

$$I_k^{\alpha} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_0^t (t-\tau)^{\frac{\alpha}{k}-1} f(\tau) d\tau \ t > 0, \tag{19}$$

where

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \ \ k > 0. \tag{20}$$

is the k-Gamma function introduced in [8] and whose relationship with the classical Gamma function is

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right). \tag{21}$$

Since the k-Gamma function is such that $\Gamma_k(\alpha) \to \Gamma(\alpha)$ when $k \to 1$, it follows that $I_k^{\alpha} \to I^{\alpha}$.

The k-integral (19) also satisfies the semi-group property

Proposition 1. *Let* $\alpha, \beta \in \mathbb{R}^+$, $f \in L^1([0,\infty))$ *and* $k > 0$, *then*

$$I_k^{\alpha} I_k^{\beta} f(t) = I_k^{\alpha+\beta} f(t) = I_k^{\beta} I_k^{\alpha} f(t). \tag{22}$$

For the proof, we remit to [17] *formula (10) p. 91.*

The left inverse operator of (19) was defined and studied by the Dorrego in [14], and it is given by the following

Definition 2.5. *Let* $k, \alpha \in \mathbb{R}^+$ *and* $n \in \mathbb{N}$ *such that* $n = [\frac{\alpha}{k}]+1$, $f \in L^1([0,\infty))$ *and* $I_k^{nk-\alpha} f(t) \in W^{n,1}[0,\infty)$; *the k-Riemann-Liouville fractional derivative is given by*

$${}_k\mathfrak{D}_{RL}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n k^n I_k^{nk-\alpha} f(t), \tag{23}$$

where $W^{n,1}[a,b]$ denotes the Sobolev space $W^{n,1}[a,b] = \{f \in L^1[a,b] : f^{(n)} \in L^1[a,b]\}$.

Remark 2.6. *If $k = 1$ (23) coincides with the classical Riemann-Liouville fractional derivative.*

Remark 2.7. *In Definitions 1 and 2 is to possible consider $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$. Therefore, in the Definition 2 must be $n = \left[\frac{\Re(\alpha)}{k}\right] + 1$.*

The results presented below can be seen in [15]

Definition 2.8. (k-Prabhakar integral) *Let $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$ and $\varphi \in L^1([0,b])$, $(0 < x < b \leq \infty)$. The k-Prabhakar integral operator is given by*

$$({}_k\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\varphi)(x) = \int_0^x \frac{(x-t)^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\alpha,\beta}[\omega(x-t)^{\frac{\alpha}{k}}]\varphi(t)dt, \quad (x > 0) \tag{24}$$
$$= \left({}_k\mathcal{E}^{\gamma}_{\alpha,\beta,\omega} * f\right)(x),$$

where

$$ {}_k\mathcal{E}^{\gamma}_{\alpha,\beta,\omega}(t) = \begin{cases} \frac{t^{\frac{\beta}{k}-1}}{k} E^{\gamma}_{k,\alpha,\beta}(\omega t^{\frac{\alpha}{k}}), & t > 0; \\ 0, & t \leq 0. \end{cases} \tag{25}$$

and $*$ mean the usual convolution product for causal functions.

Remark 2.9. *Note that if $\gamma = 0$ we have*

$$({}_k\mathbf{P}^{0}_{\alpha,\beta,\omega}\varphi)(t) = (I^{\beta}_{k}\varphi)(t) \tag{26}$$

That is, the operator (24) generalize the k-Riemann-Liouville fractional integral defined by (19).

Remark 2.10. *If we put $k = 1$ $(\gamma \neq 0)$, result*

$$({}_1\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\varphi)(t) = (\mathbf{E}^{\gamma}_{\alpha,\beta,\omega;0^+}\varphi)(t) \tag{27}$$

The following two propositions deal with the boundedness of the operator on different functions spaces.

Proposition 2. *The k-Prabhakar integral operator is bounded on $L^1([0,b])$,*
$(0 < x \leq b < \infty)$.
Let $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$ y $\varphi \in L^1([0,b])$ result

$$\|({}_k\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\varphi)(x)\|_1 \leq B\|\varphi\|_1, \tag{28}$$

where

$$B = \frac{b^{\Re(\frac{\beta}{k})}}{k}\sum_{n=0}^{\infty}\frac{|(\gamma)_{n,k}(\omega b^{\Re(\frac{\alpha}{k})})^n|}{\left[n\Re\left(\frac{\alpha}{k}\right)+\Re\left(\frac{\beta}{k}\right)\right]|\Gamma_k(\alpha n+\beta)|n!}. \tag{29}$$

Proposition 3. *The k-Prabhakar integral operator is bounded on* $C([0,x])$,
$(0 < x \le b < \infty)$.
Let $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$ *and* $\varphi \in C([0,b])$ *then we have*

$$\|({}_k\mathbf{P}^{\gamma}_{\alpha,\beta,\omega}\varphi)(x)\|_C \le B\|\varphi\|_C, \tag{30}$$

where

$$\|\varphi\|_C = \max\{|\varphi| : 0 \le x \le b\} \tag{31}$$

and B *is given by (29).*

Lemma 2.11. *Given* $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$. *Then we have*

$$I^{\alpha}_k[(t-\tau)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-\tau)^{\frac{\rho}{k}})] = (t-\tau)^{\frac{\alpha+\beta}{k}-1}E^{\gamma}_{k,\rho,\beta+\alpha}(\omega(t-\tau)^{\frac{\rho}{k}}). \tag{32}$$

Lemma 2.12. *Let* $\alpha, \beta, \omega, \gamma, \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\alpha) > 0$; $\Re(\beta) > 0$ *and* $\varphi \in L^1(\mathbb{R}^+_0)$ *and* $\left|\omega k(ks)^{-\frac{\rho}{k}}\right| < 1$. *Then the Laplace transform of the k-Prabakar integral operator is given by*

$$\mathcal{L}\{({}_k\mathbf{P}^{\gamma}_{\rho,\beta,\omega}\varphi)(x)\}(s) = \mathcal{L}\{{}_k\mathcal{E}^{\gamma}_{\rho,\beta,\omega}(t)\}(s)\mathcal{L}\{\varphi\}(s) \tag{33}$$

$$= (ks)^{-\frac{\beta}{k}}\left(1-\omega k(ks)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma}{k}}\mathcal{L}\{\varphi\}(s). \tag{34}$$

Proposition 4. *Given* $\rho, \beta, \omega, \gamma \in \mathbb{C}$, $k \in \mathbb{R}^+$ *and* $\Re(\rho) > 0, \Re(\beta) > 0, \Re(\alpha) > 0$. *Then result*

$$({}_k\mathbf{P}^{\gamma}_{\rho,\beta,\omega})[\sigma^{\frac{\alpha}{k}-1}](t) = \Gamma_k(\alpha)t^{\frac{\alpha+\beta}{k}-1}E^{\gamma}_{k,\rho,\beta+\alpha}(\omega t^{\frac{\rho}{k}}). \tag{35}$$

Proposition 5. *Given* $\rho, \beta, \mu, \delta, \omega, \gamma \in \mathbb{C}$, $k \in \mathbb{R}^+$; $\Re(\rho) > 0, \Re(\beta) > 0$ *and*
$\left|\frac{k\omega}{(ks)^{\frac{\alpha}{k}}}\right| < 1$. *Then we have*

$$({}_k\mathbf{P}^{\gamma}_{\rho,\beta,\omega})\left[\frac{\sigma^{\frac{\mu}{k}-1}}{k}E^{\delta}_{k,\rho,\mu}(\omega(x-\sigma)^{\frac{\rho}{k}})\right](t) = \frac{t^{\frac{\mu+\beta}{k}-1}}{k}E^{\delta+\gamma}_{k,\rho,\mu+\beta}(\omega(x-t)^{\frac{\rho}{k}}). \tag{36}$$

Below is how to acting the k-Prabhakar integral operator on k-Riemann-Liouville operators.

Proposition 6. *Given* $\alpha \in \mathbb{C}, (\Re(\alpha))$, *and* $\rho, \beta, \omega, \gamma \in \mathbb{C}, (\Re(\rho) > 0, \Re(\beta) > 0)$. *Then for any function* $f \in L^1([0,b]), (0 < x < b \leq \infty)$ *we have*

$$I_k^\alpha({}_k\mathbf{P}^\gamma_{\rho,\beta,\omega} f(t)) = {}_k\mathbf{P}^\gamma_{\rho,\beta+\alpha,\omega} f(t) = {}_k\mathbf{P}^\gamma_{\rho,\beta,\omega}(I_k^\alpha f(t)) \tag{37}$$

Proposition 7. *Let* $\alpha, \rho, \beta, \gamma, \omega \in \mathbb{C}$; $k \in \mathbb{R}^+$, $\Re(\alpha) > 0, \Re(\rho) > 0, \Re(\beta) > 0$; *then* $f \in L^1([0,b])$ *y* $0 < x < b \leq \infty$ *it is true that*

$${}_k\mathfrak{D}^\alpha_{RL}({}_k\mathbf{P}^\gamma_{\rho,\beta,\omega} f(t)) = {}_k\mathbf{P}^\gamma_{\rho,\beta-\alpha,\omega} f(t). \tag{38}$$

Another important property of the operator (24) is the semi-group property.

Proposition 8. *Given* $k \in \mathbb{R}^+$ *and* $\rho, \beta, \gamma, \nu, \delta, \omega \in \mathbb{C}$, $\Re(\rho) > 0, \Re(\beta) > 0, \Re(\nu) > 0$; *then, for any* $\varphi \in L^1([0,b])$ *and* $0 < x < b \leq \infty$, *we have*

$${}_k\mathbf{P}^\gamma_{\rho,\beta,\omega}({}_k\mathbf{P}^\delta_{\rho,\nu,\omega}\varphi)(t) = ({}_k\mathbf{P}^{\gamma+\delta}_{\rho,\beta+\nu,\omega}\varphi)(t) = {}_k\mathbf{P}^\delta_{\rho,\nu,\omega}({}_k\mathbf{P}^\gamma_{\rho,\beta,\omega}\varphi)(t). \tag{39}$$

As important particular case we have

$${}_k\mathbf{P}^\gamma_{\rho,\beta,\omega}({}_k\mathbf{P}^{-\gamma}_{\rho,\nu,\omega}\varphi)(t) = I_k^{\beta+\nu}\varphi(t). \tag{40}$$

Definition 2.13. (k-Prabhakar fractional derivative)(cf.[15]**)** *Given* $k \in \mathbb{R}^+$, $\rho, \beta, \gamma, \omega \in \mathbb{C}$, $\Re(\rho) > 0, \Re(\beta) > 0$, $m = \left[\frac{\beta}{k}\right] + 1$ *and* $f \in L^1([0,b])$. *We define the k-Prabhakar fractional derivative*

$${}_k\mathbf{D}^\gamma_{\rho,\beta,\omega} f(x) = \left(\frac{d}{dx}\right)^m k^m {}_k\mathbf{P}^{-\gamma}_{\rho,mk-\beta,\omega} f(x). \tag{41}$$

Remark 2.14. *If* $k = 1$ *in (41) the k-Prabhakar fractional derivative coincide with the Prabhakar fractional derivative defined in* [5].

Remark 2.15. *If* $\gamma = 0$ *in (41) the k-Prabhakar fractional derivate coincide with the k-Riemann-Liouville fractional derivative given by* [14]. *Indeed,*

$$\begin{aligned}
{}_k\mathbf{D}^0_{\rho,\beta,\omega} f(x) &= \left(\frac{d}{dx}\right)^m k^m {}_k\mathbf{P}^0_{\rho,mk-\beta,\omega} f(x) && (42)\\
&= \left(\frac{d}{dx}\right)^m k^m I_k^{mk-\beta} f(x) && (43)\\
&= {}_k\mathfrak{D}^\beta_{RL} f(x). && (44)
\end{aligned}$$

Remark 2.16. *If $k = 1$ and $\gamma = 0$, the k-Prabhakar fractional derivative coincide with the classical Riemann-Liouville fractional derivative.*

Lemma 2.17. (Laplace Transform) *The Laplace transform of k-Prabhakar fractional derivative when $[\frac{\beta}{k}] + 1 = m = 1$, is given by*

(45)

$$\mathcal{L}\{{}_k\mathbf{D}^{\gamma}_{\rho,\beta,\omega} y(x)\} = (ks)^{\frac{\beta}{k}} \left(1 - \omega k (ks)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}} \mathcal{L}\{y(x)\}(s) - k \left({}_k\mathbf{P}^{-\gamma}_{\rho,k-\beta,\omega} y\right)(0)$$

provided that $\left|\omega k (ks)^{-\frac{\rho}{k}}\right| < 1$.

3. Generalization of the fractional kinetic differintegral equation

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena. Especially, the kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and natural science. The extension and generalization of fractional kinetic equations involving many fractional operators were found [22, 27, 28, 29, 30, 31]. Motivated by a large number of the above-cited investigations on the fractional kinetic equation, in this sequel, we propose to investigate solution of a certain generalized fractional kinetic equation associated with the Prabhakar-type operator. It is also pointed out that the main results presented here can include, as their special cases, solutions of many fractional kinetic equations which are (presumably) new and known. In this section we consider an equation that generalizes the kinetic equation following the idea of the equation (5.7) given in [20].

Theorem 3.1. *Given the following Cauchy problem*

$$(1) \quad \begin{cases} a\, {}_k\mathbf{D}^{\gamma}_{p,q,\omega} N(t) - N_0 f(t) = b\, {}_k\mathbf{P}^{\delta}_{p,\nu,\omega} N(t), & f \in L^1[0,\infty); \\ \left({}_k\mathbf{P}^{-\gamma}_{\rho,k-q,\omega} N\right)(0) = c, & c \geq 0. \end{cases}$$

where $[\frac{q}{k}]+1=m=1$; $\omega\in\mathbb{C}$, $a,b\in\mathbb{R}$ $(a\neq 0)$; $p,q,k>0,\nu>0,\gamma\geq 0,\delta\geq 0$, *we have that the solution is*

$$N(t)=\frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n\left({}_k\mathbf{P}^{(\delta+\gamma)n+\gamma}_{p,(\nu+q)n+q,\omega}f\right)(t)$$

$$+\frac{c}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n t^{\frac{(\nu+q)n+q}{k}-1}E^{(\delta+\gamma)n+\gamma}_{k,p,(\nu+q)n+q}\left(\omega t^{\frac{p}{k}}\right). \tag{2}$$

Proof. Applying the Laplace transform to both sides of (1)

$$\left[\frac{a(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{\frac{\gamma+\delta}{k}}-b}{(ks)^{\frac{\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{\frac{\delta}{k}}}\right]N(s)=N_0F(s)+ck, \tag{3}$$

then

$$N(s)=\left[\frac{(ks)^{\frac{\nu}{k}}\left(1-\omega k(ks)^{-\frac{\rho}{k}}\right)^{\frac{\delta}{k}}}{a(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{\frac{\gamma+\delta}{k}}-b}\right]N_0F(s) \tag{4}$$

$$+ck\left[\frac{(ks)^{\frac{\nu}{k}}\left(1-\omega k(ks)^{-\frac{\rho}{k}}\right)^{\frac{\delta}{k}}}{a(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{\frac{\gamma+\delta}{k}}-b}\right], \tag{5}$$

$$N(s)=\left[\frac{a^{-1}(ks)^{\frac{q}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{\gamma}{k}}}{1-\frac{b}{a}(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{\gamma+\delta}{k}}}\right]N_0F(s) \tag{6}$$

$$+ck\left[\frac{a^{-1}(ks)^{\frac{q}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{\gamma}{k}}}{1-\frac{b}{a}(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{\gamma+\delta}{k}}}\right]. \tag{7}$$

Taking $\left|\frac{b}{a}(ks)^{\frac{q+\nu}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{\gamma+\delta}{k}}\right|<1$, we have

(8)
$$N(s)=\frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(ks)^{-\frac{[(q+\nu)n-q]}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{[(\gamma+\delta)n+\gamma]}{k}}F(s)$$

(9)
$$+\frac{ck}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(ks)^{-\frac{[(q+\nu)n-q]}{k}}\left(1-\omega k(ks)^{-\frac{p}{k}}\right)^{-\frac{[(\gamma+\delta)n+\gamma]}{k}}.$$

Finally, by the inverse Laplace transform, we have the desired result.

Remark 3.2. *Note that the function $N(t)$ is represented by an infinite series in terms of the k-Mittag-Leffler function (which has a close relation with the three parameters Mittag-Leffler function cf.* [9]*), therefore convergence can be tested with the same procedure used in Appendix C of* [33].

Next we obtain interesting particular cases from the Cauchy problem (1):

Corollary 3.3. *If we take $k=1$ in (1), we get the following Cauchy problem, in terms of Prabhakar derivative (cf.*[5]*) and Prabhakar integral operator (cf.*[3]*)*

(10)
$$\begin{cases} a\,\mathbf{D}^{\gamma}_{p,q,\omega}N(t)-N_0f(t)=b\,\mathbf{E}^{\delta}_{p,\nu,\omega}N(t), & f\in L^1[0,\infty);\\ \left(\mathbf{E}^{-\gamma}_{\rho,1-q,\omega}N\right)(0)=c, & c\geq 0.\end{cases}$$

where $\omega\in\mathbb{C}$, $a,b\in\mathbb{R}$ $(a\neq 0)$; $p>0$, $0<q,\nu<1,\gamma\geq 0,\delta\geq 0$, and it solution is given by

(11)
$$N(t)=\frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n\left(\mathbf{P}^{(\delta+\gamma)n+\gamma}_{p,(\nu+q)n+q,\omega}f\right)(t)$$

(12)
$$+\frac{c}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n t^{(\nu+q)n+q-1}E^{(\delta+\gamma)n+\gamma}_{p,(\nu+q)n+q}(\omega t^p).$$

Corollary 3.4. *If we take $\gamma=0$ in (1), obtain a Cauchy problem involving the k-Riemann-Lioville fractional derivative (23) and k-Prabhakar fractional integral (24)*

(13)
$$\begin{cases} {}_k\mathfrak{D}^{q}_{RL}N(t)-N_0f(t)=b\,{}_k\mathbf{P}^{\delta}_{p,\nu,\omega}N(t), & f\in L^1[0,\infty);\\ \left(I^{1-q}_{k}N\right)(0)=c, & c\geq 0.\end{cases}$$

where $[\frac{q}{k}]+1=m=1$*;* $\omega\in\mathbb{C}$, $a,b\in\mathbb{R}$ $(a\neq 0)$*;* $p,q>0,\nu>0,\gamma\geq 0,\delta\geq 0$*, we have that the solution is*

$$N(t) = \frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n\left({}_k\mathbf{P}^{\delta n}_{p,(\nu+q)n+q,\omega}f\right)(t) \tag{14}$$

$$+ \frac{c}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n t^{\frac{(\nu+q)n+q}{k}-1}E^{\delta n}_{k,p,(\nu+q)n+q}\left(\omega t^{\frac{p}{k}}\right). \tag{15}$$

Corollary 3.5. *If we take* $k=1$, $\gamma=0$, $N_0=N(0)=1$ *and* $a=1$ *in (1), obtain the Cauchy problem*

$$\begin{cases} D^qN(t)-N_0f(t)=b\,\mathbf{E}^{\delta}_{p,\nu,\omega}N(t), & f\in L^1[0,\infty); \\ \left(I^{1-q}N\right)(0)=c, & c\geq 0. \end{cases} \tag{16}$$

where $\omega\in\mathbb{C}$, $b\in\mathbb{R}$*;* $p>0$, $0<q,\nu<1$, $\delta\geq 0$*, we have that the solution is*

$$N(t)=\sum_{n=0}^{\infty}b^n\left(\mathbf{E}^{\delta n}_{p,(\nu+q)n+q,\omega}f\right)(t)+c\sum_{n=0}^{\infty}b^nt^{(\nu+q)n+q-1}E^{\delta n}_{k,p,(\nu+q)n+q}\left(\omega t^p\right). \tag{17}$$

Remark 3.6. *A similar Cauchy problem has been studied in* [16]*, with the condition* $n<q<n+1$.

Corollary 3.7. *If we take* $k=1$, $\gamma=\delta=0$, $N_0=a=1$ *and* $b=-1$ *in (1), we get the following Cauchy problem, in terms of classical Riemann-Liouville fractional operators*

$$\begin{cases} D^qN(t)+I^{\nu}N(t)=f(t), & f\in L^1[0,\infty); \\ \left(I^{1-q}N\right)(0)=c, & c\geq 0. \end{cases} \tag{18}$$

where $0<q,\nu<1$*, we have that the solution is*

$$N(t)=N_0\sum_{n=0}^{\infty}b^nI^{(\nu+q)n+q}f(t)+c\sum_{n=0}^{\infty}b^nt^{(\nu+q)n+q-1}E_{p,(\nu+q)n+q}\left(\omega t^p\right). \tag{19}$$

Remark 3.8. *The above result was obtained in* [1] *by an operational method*

In both the theorem and its corollaries, interesting examples can be considered for an appropriate choice of the function $f(t)$.

4. Solution of generalized fractional Kinetic differintegral equation by using Sumudu transform

In this section, we will investigate the solution of the generalized fractional kinetic differintegral equation (1) by applying the Sumudu transform technique.

An integral transform, called the Sumudu transform was defined and studied by G.K. Watugala [32] to facilitate the process of solving differential and integral equations in the time domain, and for the use in various applications of system engineering and applied physics. It turns out that the Sumudu transform has very special and useful properties and it is useful in solving problems of science and engineering governing kinetic equations. The Sumudu transform has been shown to be the theoretical dual of the Laplace transform.

The Sumudu transform is derived from the classical Fourier integral and defined over the set of the functions

(1)
$$A = \left\{ f\left(t\right) \Big| \exists M, \tau_1, \tau_2 > 0, \left|f\left(t\right)\right| < M\, e^{|t|/\tau_j}\ if\ t \in (-1)^j \times [0, \infty) \right\},$$

by the following formula

$$G\left(u\right) = \mathcal{S}\left\{f\left(t\right):\ u\right\} = \int_0^\infty e^{-t}\, f\left(ut\right)\,\mathrm{d}t,\ \left(-\tau_1 < u < \tau_2\right), \tag{2}$$

where M is a real finite number and τ_1 and τ_2 can be finite or infinite [32].

Hence, $G(u)$ is called as the Sumudu transform of $f(t)$. It is obvious that this is a linear operator. It can be easily verified that in (2) the function $G(u)$ keeps the same units as $f(t)$, for any real or complex number λ it gives that $S[f(\lambda t)] = G(\lambda u)$.

The Sumudu and Laplace transforms exhibit a duality relation that may be expressed either as

$$G\left(\frac{1}{u}\right) = u\,F\left(u\right) \quad \text{or} \quad G\left(u\right) = \frac{1}{u}\,F\left(\frac{1}{u}\right), \tag{3}$$

$$F\left(\frac{1}{p}\right) = p\,G\left(p\right) \quad \text{or} \quad F\left(p\right) = \frac{1}{p}\,F\left(\frac{1}{p}\right). \tag{4}$$

The convolution theorem for Sumudu transform is given by

$$S\left\{f * g;\ u\right\} = uS\left\{f; u\right\}\ S\left\{g; u\right\}, \tag{5}$$

if we apply convolution theorem for Sumudu transform, we observe that (6) can be written in the following form:

$$\mathcal{S}\left\{{}_0D_t^{-\nu}f(t)\right\} = \mathcal{S}\left\{\frac{t^{\nu-1}}{\Gamma(\nu)}\right\}.\mathcal{S}\left\{f(t)\right\} = u^{\nu}G(u)\,. \tag{6}$$

Discussion By Applying the Sumudu transform to the both sides of (1) and using the relation (6), and make use of the following formula (Sumudu inverse transform) $S^{-1}\left\{u^{\nu-1:t}\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}$, $(\min\left\{\Re(\nu), \Re(u) > 0\right\})$, and also $S^{-1}\left\{N(u):t\right\} = N(t)$, then we easily arrive the desired result (2).

5. Concluding Remarks

In this paper we have introduced an extended fractional generalization of the standard kinetic equation and established solution for the same. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation using the Prabhakar-type operators discussed in this paper contains a number of known fractional kinetic equations involving various other special functions. We also conclude this paper by remarking that the results presented here are general enough to yield, as their special cases, solutions of a number of known or new fractional kinetic equations involving other special functions.

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