MODOS DESLIZANTES APLICADOS A UN SISTEMA MACROECONOMICO

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Abstract and introduction: The causes of the inflation and how to use monetary and economic policies to stabilize this variable, has been widely analyzed by the Theory and Applied Economics. That theory results sometimes incomplete when it's translated into economic policies, for countries with high or medium persistent levels of inflation in the long run. Inflation in Latin American countries seems to be caused by different and/or additional facts than in more advanced countries.

The effects over social development and welfare of high rates of inflation are sufficiently known, but generally the main concern is the lost of purchasing power in the low income workers, the reduce of investment by the companies and the general deterioration of the Economy.

Because inflation is a phenomena that shows particularities from time to time, and from country to country, and still remains as a big unsolved problem for some Latin American countries, is needed to revisit theories, finding the basement of its mechanisms and looking for new solutions. Many hypothesis, points of view (monetary, income distribution and demand inflation), short and long run inflation, nominal rigidities and several models have been studied. Here we propose a new model for testing traditional like a modified Phillips curve, as an starting point of empirical research.

This is a project and a proposal of research, so it doesn't have results. In the first part, we explain a mathematical model that was taken from Engineering for input-output systems, which has the goal of stabilizing the output. In the second part, we expose a simple three-equations monetary model, which will be our starting point. In the third part, we explain how the research is thought to be continued.

1. THE MATHEMATICAL MODEL

SLIDING MODES

This topic have been widely analyzed in the literature, while Emilyanov and Utkin (1978, 1992) are the most classical references. In another hand, Sira Ramirez (1994) demonstrated the close relationship between Sliding Regimes and *pulse-width modulation* (Slotine and Weiping, 1991).

In this work we will use the results obtained to date and we will study its principal advantage: its immunity to input perturbations and variations in the parameters of the system.

BASIC DEFINITIONS

The Sliding Modes(SM) are part of a more general type of systems, the systems of variable structure . SM are basically a control law, which changes very quickly for conducting the path of the states of the system towards an arbitrary surface $S(x,t)$ specifically chosen by the researcher, maintaining that path over S or at least during a time interval.

The Sliding Surface will be an *attractor,* just only if certain conditions are fulfilled. The *Control in Sliding Mode* will be robust because the dynamic of the system is generated by this surface and because it will not depend on any parameter of the system.

Nevertheless, to maintain the states path over the Sliding Surface, it will be need to change the control law every time the path cuts the surface. In an ideal SM, that control action will be generated with an infinite high frequency. Because of hysteresis, lags and inertia in real systems, this frequency is finite and also can excite non modeled dynamics of the system. This phenomena is known as *chattering.*

EXISTENCE OF THE SLIDING REGIME

We take an autonomous system, expressed as:

[1] $\dot{x}_i = f_i(t, x_1, x_2, \ldots, x_n) \quad i = 1, 2, \cdots, \mathrm{n}$

where the functions $f_i(t, x_1, x_2, \ldots, x_n)$ are defined in the domain of the space of states and could be considered as discontinues functions.

Then, we can suppose that the functions $f_i(t, x_1, x_2, \ldots, x_n)$ are in general, continuous by sections and that they present discontinuity in a surface *S* defined as:

[2] $S(x_1, x_2, \cdots, x_n) = 0$

If we identify the space of states as H, then the surface will divide it into 2 regions: H^+ for $S > 0$ and H^- for $S < 0$; then in a neighborhood of *S*, functions f_i will be f_i^+ and f_i^-, defined in H^+ and H^- respectively, while f_N^+ and f_N^- are the projections over the normal N to the surface. (Figure 1)

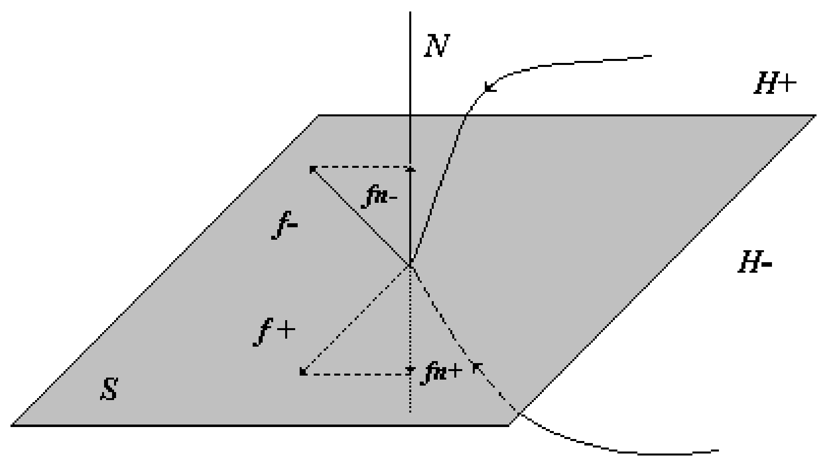

Figure 1: State Space (dimension 3) and Sliding Surface (dimension 2)

Taking derivative with respect time:

[3] $$\dot{S} = \sum_{i=1}^{n} \frac{\partial S}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^{n} \nabla S \cdot f_i(t, x_1, \cdots, x_n)$$

If Filippov conditions are fulfilled, the surface will be an attractor; in that case:

[4] $$f_N^+ < 0 \Rightarrow \dot{S} < 0$$
$$f_N^- > 0 \Rightarrow \dot{S} > 0$$

Those conditions assure that the surface will be an attractor and over it, the *sliding regime or mode* will be produced.

If the system in [1] is expressed like a controlled system *x=f(t,x,u)*, where *signal u* is discontinuous and has the form:

[5] $$u = \begin{cases} u^+(\mathrm{t, x}) & \text{for } \mathrm{S(x)} > 0 \\ u^-(\mathrm{t, x}) & \text{for } \mathrm{S(x)} < 0 \end{cases} \qquad u^+ \neq u^-$$

Then we can extend the previous analysis for an autonomous system, to a controlled one, so therefore: a *sliding regime* will happen over $S(x) = 0$, if the projections of the vectors $f^+ = f(t, x, u^+)$ and $f^-(t, x, u^-)$ over the gradient of the surface S, are opposed signs and are directed towards the surface. Analytically:

[6] $\lim_{S\to 0+} \dot{S} < 0$ and $\lim_{S\to 0-} \dot{S} > 0$

GENERAL ASPECTS OF A CONTROLLED SYSTEM IN A SLIDING REGIME

We consider a system of one entry, expressed in its Controllable Canonic Form (CCF) by its dynamic equation:

[7] $\dot{x} = f(\mathbf{x}) + b(\mathbf{x}) \cdot u$

where the scalar x is the output in which we are interested, $\mathbf{x}$ is the vector of states and scalar u is the control signal. About the mostly linear and not exactly known function $f(\mathbf{x})$, it can be assumed that it's limited by a known function of $\mathbf{x}$. Also, the control gain $b(\mathbf{x})$ is not exactly known, but its sign is known and it´s limited.

The control problem is about state $\mathbf{x}$, to achieve or follow a desired state, which could vary across time ($\mathbf{x}_d = [x_d, \dot{x}_d, \cdots, x_d^{n-1}]^{\mathsf{T}}$), even in front of inaccuracies of the model produced by $f(\mathbf{x})$ and $b(\mathbf{x})$. For the achievement of the previous, using a finite control signal u, the initial state desired $\mathbf{x}_d(0)$ must be:

[8] $\mathbf{x}_d(0) = \mathbf{x}(0)$

If $\tilde{x} = x - x_d$ is the error in the variable x, and $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d = [\tilde{x}, \dot{\tilde{x}}, \cdots, \tilde{x}^{(n-1)}]^{\mathsf{T}}$ is the error vector, then it can be defined a time-variant surface $S(t)$ in the space of states $\Re^{(n)}$ by the scalar equation $s(\mathbf{x},t) = 0$ where:

[9] $s(\mathbf{x},t) = \left(\frac{d}{dt} + c\right)^{n-1} \tilde{x}$ with $c > 0$

For example, if *n=3* then $s = \ddot{x} + 2 \cdot c \cdot \dot{x} + c^2 \cdot x$.

Then, regarding the initial conditions [8], the problem of achieving $\mathbf{x} \equiv \mathbf{x}_d$ is equivalent to get the error vector $\tilde{\mathbf{x}}$ to stay over the surface *S(t)* for every t>0. The problem of tracing the n-dimensional vector $\mathbf{x}_d$ can be replaced by a first order stabilization problem of *s*. In fact, because the expression of *s* embedded in [9] contains the element $\tilde{x}^{(n-1)}$, just deriving *s* one time, it appears the entrance signal u.

The simplified first order problem (to bring scalar *s* to cero), can be solved picking control law u from [7] such that the condition for the existence of a sliding mode exists.

ROBUSTNESS OF THE SLIDING MODE SYSTEMS

We consider a system expressed in its Controllable Canonic Form:

$$[10]\quad \begin{aligned} &\dot{x} = x_{i+1} \qquad \mathrm{i} = 1,2,\ldots,\mathrm{n}-1 \\ &\vdots \\ &\dot{\mathrm{x}}_n = -\sum_{i=1}^{n}\delta_i \cdot x_i + \beta(t) + u \end{aligned}$$

where u is the control signal; $\beta(\mathrm{t})$ is an entrance or non structural perturbation, in the sense that is not originated by the parameters defined in the system; and δ_i are system parameters, that can be assumed constant or variant in time.

We can assume that δ_i and $\beta(\mathrm{t})$ are unknown, and that control signal u is of the type defined in [5].

If we define the *Sliding Surface S(x)* like:

$$[11]\quad S(x) = \sum_{i=1}^{n} c_i \cdot x_i \quad c_i = const \quad c_n = 1$$

where c_i is a constant and $c_n = 1$, and taking into account that we can assign an arbitrary dynamic to *S(x)*, then the conditions for the existence of a *Sliding Mode* in *S(x)=0* can be fullfilled, which were pointed out in [6].

Then, if [6] is fulfilled and equation [11] is solved for *S(x)=0* and for the variable x_n, the following is obtained:

$$[12]\quad x_n = -\sum_{i=1}^{n-1} c_i \cdot x_i \qquad \text{and} \qquad \begin{aligned} &\dot{x}_i = x_{i+1} \qquad \mathrm{i} = 1,2,\ldots\ \mathrm{n}-2 \\ &\vdots \\ &\dot{\mathrm{x}}_{n-1} = -\sum_{i=1}^{n-1} c_i \cdot x_i \end{aligned}$$

The latter result performs the set of equations from which the *SM* is going to depend. It can be observed that nor the perturbation nor the unknown parameters appear; the only influential are the parameters c_i of the *Sliding Surface*, which are constants of the design.

Example 1: Let's consider a system of two state variables. The perturbation is modeled as a function $F_p = a\sin(wt)$. The model in state variables is:

$$[13] \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & -K2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_p$$

and then a *Sliding Surface* can be defined in agree with [11]:

[14] $S(t) = x_2(t) + c_1 \cdot x_1(t) = 0$ where c_1 is a design constant.

As we said before, this sliding surface must behave as an "attractor" and will be a line in the phase plane by which the states of the system will "slide" towards origin.

The abovementioned will be fulfilled if the Liapunov function: $L = \frac{1}{2}S^2$ is defined, then $\dot{L} = S \cdot \dot{S} < 0$ and could be easily satisfied if the *sign* function is used:

[15] $\dot{S} = -\eta \cdot sign(S)$, where η is a positive design constant, and *sign* the function:

$$[16] \quad sign(S) = \begin{cases} 1 & \forall S > 0 \\ 0 & \forall S = 0 \\ -1 & \forall S < 0 \end{cases}$$

Combining [14] and [15]:

$$[17] \quad \dot{x}_2 + c_1 \cdot \dot{x}_1 = -\eta \cdot sign(S)$$

And relating [14] and [17] we get the final expression of the control:

$$[18] \quad u_{ev} = \frac{1}{K_3}\left[K_1 \cdot x_1 + (K_2 - c_1) \cdot x_2 - Fp - \eta \cdot sign(S)\right]$$

In the latter, c_i and η are parameters which are part of the controller's design; the best performance of the system will depend on an accurate choice of them. The simulation can be watched in figures 2 and 3, where the values used in the controller where: $c_i = 3$ and η =0.05.

As expected, the system responded well to the control in sliding mode; however, is clear that control signal is inapplicable in practice, due to the presence of *"chattering"*.

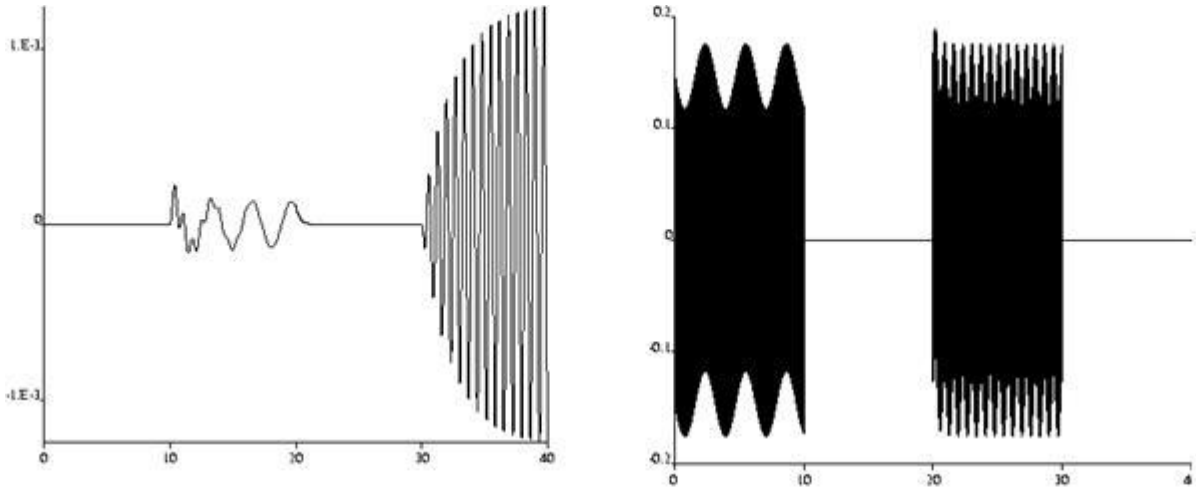

Fig. 2: a) Output of the system with and without control, at different frequencies b) Control signal applied.

The signal in we are interested can be observed with and without control in the figure 2 a) and so the control signal applied, in b). When the control is applied, the output is zero, while when no control is applied, the output begin to be affected by the perturbation.

The presence of *chattering* is undesirable in the practice, because it involves very high frequency oscillations that can excite non modeled dynamics, low precision in the control. Then, is required to *smooth* the discontinuous control signal to achieve an engagement between the signal tolerated by the real physical elements and the required precision.

In a Macroeconomic model, where sample time could be hourly, daily and highly frequencies (not milliseconds, like in Engineering), this can be solved more easily. Chattering can be eliminated or reduced a lot, with a boundary layer.

2. THE MACROECONOMIC MODEL

The Macroeconomic model is a modified Phillips curve, expressed in differential equations and trying to find an expression for the dynamic behaviour of the rate of inflation.

The Phillips curve speaks about the trade-off between inflation and unemployment. Such a relationship can look as too much simple, but it will be a good point for beginning the modelling. In the future, this simple 3-equations-model can be extended.

Let's think about just only 2 markets in the Economy, labour market and money market. If inflation rate is a basically a wages function, we can write:

[19] $p = \alpha - \beta U - T$; α y β >0

where: p is the inflation rate (if P is the level price, then $p = \dot{P}/P$, U is the unemployment rate and T is the work productivity (which can vary with t) and wages are a lineal function of unemployment, like w=$\alpha - \beta U$ in [19].

If we add the fact that today, economic agents have an expected level of inflation for next period, then:

[20] $p = \alpha - \beta U - T + h\pi$; $0 < h \leq 1$ and π is the level of inflation expected by the agents.

Inflation expectations could be modeled as *adaptative expectations*, like:

[21] $\frac{d\pi}{dt} = j(p - \pi)$; where t is time and $0 < j \leq 1$

If we add a third equation, then [20] [21] and [22] will be our three equations system:

[22] $\frac{dU}{dt} = -k(m - p)$; with k> 0 and where m is the rate of growth of money M ($\dot{M}/M$).

The right side of [22] is the rate of growth of real money. Therefore, this equation specifies a negative relationship between unemployment variation and real money growth. It also shows a feedback from inflation variable to unemployment variable.

Then, our model of three endogenous variables $\pi, p\ and\ U$ is the set of equations [20], [21]and [22].

Time path of inflation rate

Now we will resume our three equations model in one, by substituting [20]in[21], expressing the whole system for explaining behaviour of the rate of the expected level of inflation:

[23] $\frac{d\pi}{t} = j(\alpha - T - \beta U) - j(1 - h)\pi$

We take differential of [23] respect to t:

[24] $\frac{d^2\pi}{dt^2} = -j\beta\frac{dU}{dt} - j(1-h)\frac{d\pi}{dt}$

Substituting [22]in[24]:

[25] $\frac{d^2\pi}{dt^2} = j\beta km - j\beta kp - j(1-h)\frac{d\pi}{dt}$

We clear *p* from [21] and replace in [25], obtaining:

[26] $\frac{d^2\pi}{dt^2} + \underbrace{[\beta k + j(1-h)]}_{a_1}\frac{d\pi}{dt} + \underbrace{(j\beta k)}_{a_2}\pi = \underbrace{j\beta km}_{b}$

In this way, the particular integral of [26] is:

[27] $\pi_p = \frac{b}{a_2} = m$: the intertemporal equilibrium value of the rate of inflation, depending just only in the rate of growth of **nominal** money.

The complementary function will be:

[28] $r_1.r_2 = \frac{1}{2}(-a_1 \pm \sqrt{a_1^2 - 4a_2})$

So, we were interested in obtaining and expression for π and its dynamic in time. *A priori,* we just can say that because of [26], a_1 and a_2 are positive constants, but we can't know if they are equal or different. Then [28] can result in different real roots, equal or complex. For different and equal real roots, we can deduct that both roots will be negative, so it will exist a dynamic stable equilibrium.

3. FURTHER WORK

This simple Phillips curve is just the starting point to test the usefulness of the mathematical model. After that, the research will follow:

- Elaborating a bigger system of equations.
- Testing other theories besides Phillips curve, including short-run behaviour.

- Using Econometrics to feed the math model, by the estimation of parameters. It will be used multivariate ARIMA models.
- Another Econometric techniques could be used, like Bayesian estimators.
- Inflation Theories are going to be tested first in developed countries like United States, France and Germany. The former will be done to have a comparison point with Argentina, Brazil, and Venezuela.
- The main goal will be testing the existence of more complex combination of causes in Latin American countries of actual persistent inflation, than in high-income countries.

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