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The Generalized k-α-Miller-Ross Function

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Abstract

The aim of this short paper is to introduce the ${}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda)$ function given by the following series

$$
{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda) = (kt)^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}(\lambda t^{\frac{\alpha}{k}})^n t^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha n+\alpha+\nu)(n+j)!},
$$

where $(\gamma)_{n,k}$ is the Pochhammer k-symbol and $\Gamma_k(z)$ is the k-Gamma function. This function is a generalization of the Miller-Ross function $E_t(\nu,\lambda)$ and of the k-α-Exponential function ${}_k e^{\lambda z}_{\gamma,\alpha}$. We study various properties of the ${}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda)$ function including its fractional derivative.

Mathematics Subject Classification: 33xx, 26A33

Keywords: Fractional Calculus, Miller-Ross function, Mittag-Leffler function

1 Introduction and preliminary notes

As is well know, cf.[7] the Miller-Ross function $E_t(\nu,a)$ is defined as the ν-th integral of the exponential function i.e.

$$
E_t(\nu,\lambda) = I^{\nu}e^{\lambda t} = t^{\nu}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{\Gamma(n+\nu+1)}, \tag{1}
$$

where ν and λ are real numbers, $\nu > 0$, and I^ν is the Riemann-Liouville fractional integral of order ν given by the following expression

$$I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x) dx, \tag{2}$$

where $\Gamma(\nu)$ is the classical Gamma function.

Let α and β be real numbers, the two parameters Mittag-Leffler function $E_{\alpha,\beta}(t)$ is given by

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0,\ \beta > 0. \tag{3}$$

If $\beta = 1$, from (3) we obtain the one parameter Mittag-Leffler function

$$E_{\alpha,1}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \tag{4}$$

From (1) and (3), the Miller-Ross function may be written as

$$E_t(\nu, \lambda) = t^\nu E_{1,1+\nu}(\lambda t), \tag{5}$$

where in the right hand member $E_{1,1+\nu}(\lambda t)$ is the Mittag-Leffler of two parameters when $\alpha = 1$, and $\beta = \nu + 1$.

Other generalization of the Mittag-Leffler function is the one introduced by us in [6] given by

$${}_kE^{\gamma}_{\alpha,\beta}(\lambda t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{(\lambda t)^n}{n!}, \quad \alpha > 0, \tag{6}$$

where $(\gamma)_{n,k}$ is the Pochhammer k-symbol (cf.[5])

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k)...(\gamma + (n-1)k),$$

and $\Gamma_k(z)$ denotes the k-Gamma function (cf.[5]), a one parameter deformation of the classical Gamma function, that admit the following integral representation

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt,\ k \in \mathbb{R}^+,\ z \in \mathbb{C}. \tag{7}$$

Let α and γ be complex numbers that $\Re(\alpha) > 0$, $\Re(\gamma) > 0$ and $k > 0$. The k-α-Exponential function is defined (cf.[3]) as

$${}_ke^{\lambda z}_{\gamma,\alpha} = z^{\frac{\alpha}{k}-1} {}_kE^{\gamma}_{\alpha,\alpha}(\lambda z^{\frac{\alpha}{k}}) \tag{8}$$

where ${}_kE^{\gamma}_{\alpha,\alpha}(\lambda z^{\frac{\alpha}{k}})$ is the k-Mittag-Leffler function given in (6).

In [2] formula (VI.1) we have introduced a new family of Mittag-Leffler type functions obtained by a slight modification of the k-α-Exponential function adding one more parameter. We have the family of functions

$$ {}_k\mathcal{E}^{k,\gamma,\alpha}_{j}(\lambda z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\, \lambda^n z^{\frac{\alpha}{k}(n+1)-1}}{\Gamma_k(\alpha(n+1))(n+j)!} \tag{9} $$

where α, γ are complex numbers such that $\Re(\alpha) > 0$, $\Re(\gamma) > 0$, $k > 0$ and $z \in \mathbb{C} - \{0\}$, and $j \in \mathbb{N}_0$. It can easy be seen that ${}_k\mathcal{E}^{k,\gamma,\alpha}_{j}(\lambda z)$ reduces to the k-α-Exponential function given by (8) when $j = 0$ is considered.

For the further development of this paper, we will need certain properties of the functions ${}_k\mathcal{E}^{k,\gamma,\alpha}_{j}(\lambda z)$, which will be summarized in the following

Lemma 1.1 *Let α and γ be complex numbers such that $\Re(\alpha) > 0$, $\Re(\gamma) >$ 0, $k > 0$ and $z \in \mathbb{C} - \{0\}$, $j \in \mathbb{N}_0$. Then:*

1. $D^{\frac{\alpha}{k}} \left[{}_k\mathcal{E}^{k,\gamma,\alpha}_{j}(\lambda z) \right] = \lambda k^{-\frac{\alpha}{k}}\, {}_k\mathcal{E}^{k,\gamma,\alpha}_{j+1}(\lambda z),\ j \in \mathbb{N}_0$

2. $\left(D^{\frac{\alpha}{k}}\right)^j \left({}_ke^{\lambda z}_{\gamma,\alpha}\right) = \left(\lambda k^{-\frac{\alpha}{k}}\right)^j {}_k\mathcal{E}^{k,\gamma,\alpha}_{j}(\lambda z)$

For the proof we remit to [2]. We also need the following

Definition 1.2 *The Riemann-Liouville integral of order ν of a function f is given by*

$$ I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau; \tag{10} $$

$t \in \mathbb{R}^+, \nu \in \mathbb{C}; \Re(\nu) > 0.$

Definition 1.3 *The Riemann-Liouville fractional derivative of order ν of a function f is given by*

$$ D^{\nu} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\nu} f(t);\ t \in \mathbb{R}^+,\ n = [\Re(\nu)] + 1. \tag{11} $$

We also note the relationship between the k-Gamma function and the Pochhammer k-symbol given by the following

Proposition 1.4 *Given $z \in \mathbb{C} \setminus k\mathbb{Z}; k, s > 0$ and $n \in \mathbb{N}$, the following identity holds*

1.
$$ (z)_{n,s} = \left(\frac{s}{k}\right)^n \left(\frac{kz}{s}\right)_{n,k} \tag{12} $$

2.
$$ \Gamma_s(z) = \left(\frac{s}{k}\right)^{\frac{z}{k}-1} \Gamma_k\left(\frac{kz}{s}\right) \tag{13} $$

For the proof we remit to Proposition 4 in [5].

2 The generalized k-α-Miller-Ross function and some of its properties

Based on the expression (1) that allows us to write the Miller-Ross function as a Mittag-Leffler type function, and by using (9) which contains as particular case the classical Mittag-Leffler function, we have the following

Definition 2.1 *Let α and ν be complex numbers such that $\Re(\alpha) > 0$, $\Re(\nu) > 0$, and $k > 0$. The k-α-generalized Miller-Ross function is given by*

$$ {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda) = t^{\frac{\nu}{k}} k^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k} \left(\lambda t^{\frac{\alpha}{k}}\right)^n t^{\frac{\alpha}{k}-1}}{\Gamma_k\left[\alpha n + (\alpha+\nu)\right](n+j)!} \tag{14} $$

Easily it can be seen that when $\alpha = 1$, $k = 1$, $\gamma = 1$, (14) coincides with (1).

Also, it can be seen that (14) may be expressed as the $\frac{\nu}{k}$−integral of the ${}_k\mathcal{E}_j^{k,\alpha,\alpha}$ function. In fact

$$
\begin{aligned}
I^{\frac{\nu}{k}}\left[{}_k\mathcal{E}_j^{k,\alpha,\alpha}\right](t) &= I^{\frac{\nu}{k}}\left[\sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}(n+1)-1}}{\Gamma_k\left[\alpha(n+1)\right](n+j)!}\right](t) \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n \Gamma\left(\frac{\alpha(n+1)}{k}\right) t^{\frac{\alpha}{k}(n+1)+\frac{\nu}{k}-1}}{\Gamma_k\left[\alpha(n+1)\right]\Gamma\left(\frac{\alpha(n+1)}{k}+\frac{\gamma}{k}\right)(n+j)!} \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n k^{1-\frac{\alpha(n+1)}{k}} \Gamma_k\left(\alpha(n+1)\right) t^{\frac{\alpha}{k}(n+1)+\frac{\nu}{k}-1}}{\Gamma_k\left[\alpha(n+1)\right]\Gamma\left(\frac{\alpha(n+1)}{k}+\frac{\gamma}{k}\right)(n+j)!} \\
&= \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n k^{1-\frac{\alpha(n+1)}{k}} t^{\frac{\alpha}{k}(n+1)+\frac{\nu}{k}-1}}{k^{1-\frac{\alpha(n+1)+\nu}{k}}\Gamma_k\left[\alpha(n+1)+\nu\right](n+j)!} \\
&= k^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}n+\left(\frac{\alpha+\nu}{k}-1\right)}}{\Gamma_k\left[\alpha(n+1)+\nu\right](n+j)!} \\
&= k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{\Gamma_k\left[\alpha n+(\alpha+\nu)\right](n+j)!} \\
&= {}_{k,j}\mathbb{E}^{\alpha,\gamma}(\nu,\lambda)
\end{aligned}
$$

2.1 Special cases of the function ${}_{k,j}\mathbb{E}^{\alpha,\gamma}(\nu,\lambda)$.

In this paragraph we will show that the function given by (14) fulfills to analogous properties verified by the classic Miller-Ross function.

For particular choices of the values of the parameters, we have the following cases:

1. When $j = 0$, we have

$$ {}_{k,0}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda) = k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu}{k}-1} {}_kE^{\gamma}_{\alpha,\alpha+\nu}(\lambda t^{\frac{\alpha}{k}}), \tag{15} $$

   where ${}_kE^{\gamma}_{\alpha,\alpha+\nu}(\lambda t^{\frac{\alpha}{k}})$ is the k-Mittag-Leffler function given by (6).

2. When $k = \nu = \alpha = 1$, we obtain

$$ E_t(\nu,\lambda) = t^{\nu} E_{1,\nu+1}(\lambda t) \tag{16} $$

   that is the formula (E.36) from [7].

3. For $\nu = j = 0$

$$ \begin{aligned} {}_{k,0}\mathbb{E}^{\alpha,\gamma}(0,\lambda) &= t^{\frac{\alpha}{k}-1} {}_kE^{0,\gamma}_{\alpha,\alpha}(\lambda t^{\frac{\alpha}{k}}) && (17)\\ &= t^{\frac{\alpha}{k}-1} {}_kE^{\gamma}_{\alpha,\alpha}(\lambda t^{\frac{\alpha}{k}}) && (18)\\ &= {}_ke^{\lambda t}_{\gamma,\alpha} && (19) \end{aligned} $$

   which coincides with the k-α-Exponential function given by (8).

4. When $t = 0$,

$$ {}_{k,j}\mathbb{E}_0^{\alpha,\gamma}(\nu,\lambda) = 0 \tag{20} $$

5. When $\lambda = 0$,

$$ {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,0) = \frac{k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu}{k}-1}}{\Gamma_k(\alpha+\nu) j!} \tag{21} $$

6. When $k = \alpha = \gamma = 1$; $j = \lambda = 0$

$$ {}_{1,0}\mathbb{E}_t^{1,1}(\nu,0) = \frac{t^{\nu}}{\Gamma(\nu+1)} \tag{22} $$

   which coincides with (E.41) from [7] and formula in page 70 of [4].

7. If $\nu = -\alpha$, we have

$$ {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-\alpha,\lambda) = \lambda k^{-\frac{\alpha}{k}} {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda) \tag{23} $$

   In fact

$$ {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-\alpha,\lambda) = k^{-\frac{\alpha}{k}} t^{\frac{\alpha-\alpha}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}(\lambda t^{\frac{\alpha}{k}})^n}{\Gamma_k(\alpha n)(n+j)!} $$

$$
\begin{aligned}
&= k^{-\frac{\alpha}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}n-1}}{\Gamma_k(\alpha n)(n+j)!} \\
&= k^{-\frac{\alpha}{k}} \sum_{m=-1}^{\infty} \frac{(\gamma)_{m+1+j,k}\lambda^{m+1} t^{\frac{\alpha}{k}(m+1)-1}}{\Gamma_k(\alpha(m+1))(m+1+j)!} \\
&= k^{-\frac{\alpha}{k}} \sum_{m=-1}^{\infty} \frac{(\gamma)_{m+(j+1),k}\lambda^{m+1} t^{\frac{\alpha}{k}m} t^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha m+\alpha)\,[m+(j+1)]!} \\
&= k^{-\frac{\alpha}{k}} t^{\frac{\alpha}{k}-1}\lambda \sum_{m=-1}^{\infty} \frac{(\gamma)_{m+(j+1),k}(\lambda t^{\frac{\alpha}{k}})^m}{\Gamma_k(\alpha m+\alpha)\,[m+(j+1)]!} \\
&= k^{-\frac{\alpha}{k}} t^{\frac{\alpha+0}{k}-1}\lambda \sum_{m=-1}^{\infty} \frac{(\gamma)_{m+(j+1),k}(\lambda t^{\frac{\alpha}{k}})^m}{\Gamma_k(\alpha m+(\alpha+0))\,[m+(j+1)]!} \\
&= \lambda k^{-\frac{\alpha}{k}}\, {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda)
\end{aligned}
$$

Remark 2.2 *When $\alpha = \gamma = k = 1$; and $j = 0$ (23) reduces to*

$$
{}_{1,0}\mathbb{E}_t^{1,1}(-1,\lambda) = \lambda\, {}_{1,0}\mathbb{E}_t^{1,1}(0,\lambda),
$$

i.e.

$$
E_t(-1,\lambda) = \lambda E_t(0,\lambda), \tag{24}
$$

cf. [1], 3.3, pp.49 ; (E.39) from [7], and in page 69 from [4].

8. If $\nu = -p\,\alpha$, it has

$$
{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-p\alpha,\lambda) = \lambda^p k^{-\frac{p\alpha}{k}}\, {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda), \quad p = 0,1,2,... \tag{25}
$$

Proof.

$$
\begin{aligned}
{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-p\alpha,\lambda) &= k^{-\frac{p\alpha}{k}} t^{\frac{\alpha-p\alpha}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{\Gamma(\alpha n+\alpha-p\alpha)(n+j)!} \\
&= k^{-\frac{p\alpha}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}(n+1-p)-1}}{\Gamma(\alpha(n+1-p))(n+j)!} \\
&= k^{-\frac{p\alpha}{k}} \sum_{m=-p}^{\infty} \frac{(\gamma)_{m+p+j,k}\lambda^{m+p} t^{\frac{\alpha}{k}(m+1)-1}}{\Gamma(\alpha(m+1))(m+p+j)!} \\
&= k^{-\frac{p\alpha}{k}} \lambda^p t^{\frac{\alpha}{k}-1} \sum_{m=0}^{\infty} \frac{(\gamma)_{m+p+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^m}{\Gamma(\alpha m+\alpha)(m+p+j)!} \\
&= \lambda^p k^{-\frac{p\alpha}{k}}\, {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda)
\end{aligned}
$$

Remark 2.3 *When $\alpha = \gamma = k = 1$, and $j = 0$, (25) reduces at*

$$E_t(-p, \lambda) = \lambda^p E_t(0, \lambda), \tag{26}$$

cf. [1], 3.4, pp. 49 and formula in page 70 of [4].

2.2 Differentiation

In this paragraph we will show some results of differentiation of the generalized k-α-Miller-Ross function which generalizes ones already known, for example those shown in [4] and in [9].

Lemma 2.4

$$\frac{d}{dt}\left({}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda)\right) = {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu - k, \lambda) \tag{27}$$

Proof

$$\begin{aligned}
\frac{d}{dt}\left({}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda)\right) &= k^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n \left(\frac{\alpha}{k}n + \frac{\alpha+\nu}{k} - 1\right) t^{\frac{\alpha}{k}n + \frac{\alpha+\nu}{k} - 2}}{\Gamma_k\left(\alpha n + \alpha + \nu\right)(n+j)!} \\
&= k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu-k}{k} - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\frac{\alpha n + \alpha + \nu - k}{k}\right)\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{k^{\frac{\alpha n + \alpha + \nu}{k} - 1}\Gamma\left(\frac{\alpha n + \alpha + \nu - k + k}{k}\right)(n+j)!} \\
&= k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu-k}{k} - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\frac{\alpha n + \alpha + \nu - k}{k}\right)\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{k^{\frac{\alpha n + \alpha + \nu}{k} - 1}\left(\frac{\alpha n + \alpha + \nu - k}{k}\right)\Gamma\left(\frac{\alpha n + \alpha + \nu - k}{k}\right)(n+j)!} \\
&= k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu-k}{k} - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{k\Gamma_k\left(\alpha n + \alpha + (\nu - k)\right)(n+j)!} \\
&= k^{\frac{\nu-k}{k}} t^{\frac{\alpha+(\nu-k)}{k} - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{k\Gamma_k\left(\alpha n + \alpha + (\nu - k)\right)(n+j)!} \\
&= {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu - k, \lambda)
\end{aligned}$$

When $\alpha = \gamma = k = 1$, and $j = 0$ (27) reduces at

$$\frac{d}{dt}\left({}_{1,0}\mathbb{E}_t^{1,1}(\nu,\lambda)\right) = {}_{1,0}\mathbb{E}_t^{1,1}(\nu - 1, \lambda) \tag{28}$$

i.e.

$$\frac{d}{dt}\left(E_t(\nu,\lambda)\right) = E_t(\nu - 1, \lambda) \tag{29}$$

which coincides with formula (14), of Theorem 3 by Susumu Sakabibara cf[9].

Also, it can be shown that when $\nu = 1$, from (29) it results

$$\frac{d}{dt}\left(E_t(1,\lambda)\right) = E_t(0, \lambda). \tag{30}$$

2.2.1 Fractional derivative of order $\frac{\mu}{k}$

Lemma 2.5

$$D^{\frac{\mu}{k}}\,{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda) = {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu-\mu,\lambda) \tag{31}$$

Proof

$$\begin{aligned}
D^{\frac{\mu}{k}}\,{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu,\lambda) &= k^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n D^{\frac{\mu}{k}}\left(t^{\frac{\alpha}{k}n+\frac{\alpha+\nu}{k}-1}\right)}{\Gamma_k\left(\alpha n+(\alpha+\nu)\right)(n+j)!} \\
&= k^{\frac{\nu}{k}} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n \Gamma\left(\frac{\alpha n+\alpha+\nu}{k}\right) t^{\frac{\alpha}{k}n+\frac{\alpha+\nu}{k}-\frac{\mu}{k}-1}}{\Gamma_k\left(\alpha n+(\alpha+\nu)\right)\Gamma\left(\frac{\alpha n+\alpha+\nu}{k}-\frac{\mu}{k}\right)(n+j)!} \\
&= k^{\frac{\nu}{k}} t^{\frac{\alpha+\nu-\mu}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}n}}{k^{\frac{\mu}{k}}\Gamma_k\left(\alpha n+(\alpha+\nu-\mu)\right)(n+j)!} \\
&= k^{\frac{\nu-\mu}{k}} t^{\frac{\alpha+\nu-\mu}{k}-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}n}}{\Gamma_k\left(\alpha n+(\alpha+\nu-\mu)\right)(n+j)!} \\
&= {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu-\mu,\lambda)
\end{aligned} \tag{32}$$

From (32), if $k=\alpha=\gamma=1$, and $j=0$, we have

$$D^{\mu}[{}_{1,1}\mathbb{E}_t^{1,1}(\nu,\lambda)] = {}_{1,1}\mathbb{E}_t^{1,1}(\nu-\mu,\lambda), \tag{33}$$

or equivalentely

$$D^{\mu}E_t(\nu,\lambda) = E_t(\nu-\mu,\lambda) \tag{34}$$

Corollary 2.6 *If $\mu=\alpha$ is considered, from (32) we have*

$$D^{\frac{\alpha}{k}}[{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda)] = {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-\alpha,\lambda), \tag{35}$$

and from (23)

$${}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-\alpha,\lambda) = \lambda k^{-\frac{\alpha}{k}}\,{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda). \tag{36}$$

Then, it results

$$D^{\frac{\alpha}{k}}[{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda)] = {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(-\alpha,\lambda) = \lambda k^{-\frac{\alpha}{k}}\,{}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(0,\lambda) \tag{37}$$

If in (37) $k=\alpha=\gamma=1$, $j=0$ is considered, we have

$$DE_t(0,\lambda) = E_t(-1,\lambda) = \lambda E_t(0,\lambda) \tag{38}$$

2.3 Integration of k-α-Miller-Ross function.

Lemma 2.7

$$\int_0^t {}_{k,j}\mathbb{E}_u^{\alpha,\gamma}(\nu,\lambda)du = {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu+k,\lambda), \qquad \Re(\nu) > -k. \tag{39}$$

Proof.

Taking into account that

$$\int_0^t u^{\frac{\alpha}{k}n+\frac{\alpha+\nu}{k}-1}du = \frac{t^{\frac{\alpha}{k}n+\frac{\alpha+\nu}{k}}}{\frac{\alpha}{k}n+\frac{\alpha+\nu}{k}} \tag{40}$$

we get

$$\int_0^t {}_{k,j}\mathbb{E}_u^{\alpha,\gamma}(\nu,\lambda)du = k^{\frac{\nu}{k}}t^{(\frac{\alpha+\nu}{k}+1)-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n+j,k}\lambda^n t^{\frac{\alpha}{k}n}}{\Gamma_k\left(\alpha n+(\alpha+\nu)\right)\left(\frac{\alpha n+\alpha+\nu}{k}\right)(n+j)!} \tag{41}$$

But

$$\Gamma_k\left(\alpha n+(\alpha+\nu)\right) = k^{\frac{\alpha n+\alpha+\nu}{k}-1}\Gamma\left(\frac{\alpha n+(\alpha+\nu)}{k}\right); \tag{42}$$

Then

$$\Gamma_k\left(\alpha n+(\alpha+\nu)\right)\left(\frac{\alpha n+\alpha+\nu}{k}\right) =$$

$$= k^{\frac{\alpha n+\alpha+\nu}{k}-1}\Gamma\left(\frac{\alpha n+(\alpha+\nu)}{k}\right)\left(\frac{\alpha n+\alpha+\nu}{k}\right) \tag{43}$$

$$= k^{\frac{\alpha n+\alpha+\nu}{k}-1}\Gamma\left(\frac{\alpha n+(\alpha+\nu)}{k}+1\right) \tag{44}$$

$$= k^{\frac{\alpha n+\alpha+\nu}{k}-1}k^{1-\frac{\alpha n+\alpha+\nu+k}{k}}\Gamma_k\left(\alpha n+(\alpha+\nu+k)\right) \tag{45}$$

$$= k^{-1}\Gamma_k\left(\alpha n+(\alpha+\nu+k)\right) \tag{46}$$

Then

$$\begin{aligned}\int_0^t {}_{k,j}\mathbb{E}_u^{\alpha,\gamma}(\nu,\lambda)du &= k^{\frac{\nu+k}{k}}t^{\frac{\alpha+\nu+k}{k}-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{\Gamma_k(\alpha n+\alpha+\nu+k)(n+j)!}\\ &= {}_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\nu+k,\lambda)\end{aligned} \tag{47}$$

When $k=j=\alpha=\gamma=1$, (47) coincides with [4].

Lemma 2.8 *If* $\Re(w) > -k,$ *and* $\Re(\nu) > -k$

$$\int_0^t u^w {}_{k,j}\mathbb{E}_{t-u}^{\alpha,\gamma}(\nu,\lambda)du = \Gamma_k(kw+k)_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\gamma+kw+k,\lambda) \tag{48}$$

Proof.

Let consider the integral

$$\int_0^t u^w (t-u)^{\frac{\alpha+\nu}{k}-1+\frac{\alpha}{k}n} du = \underbrace{\int_0^t u^{(w+1)-1} (t-u)^{\frac{\alpha+\nu+\alpha n}{k}-1} du}_{*} \tag{49}$$

by calling: $w+1=x$, and $\frac{\alpha+\nu+\alpha n}{k}=y$, then

$$* = \underbrace{\int_0^t u^{x-1}(t-u)^{y-1} du}_{**} \tag{50}$$

making the change of variables: $\frac{u}{t}=T$, $du=tdT$

$$\begin{aligned}
** &= \int_0^t (tT)^{x-1} t^{y-1} (1-T)^{y-1} t dt && (51)\\
&= t^{x+y-1} \int_0^1 T^{x-1}(1-T)^{y-1} dT && (52)\\
&= t^{x+y-1} B(x,y) && (53)\\
&= t^{x+y-1} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} && (54)\\
&= t^{w+1+\frac{\alpha+\nu+\alpha n}{k}} \frac{\Gamma(w+1)\Gamma\left(\frac{\alpha+\gamma+\alpha n}{k}\right)}{\Gamma(w+1+\frac{\alpha+\nu+\alpha n}{k})} && (55)\\
&= \frac{k^{w+1}\Gamma(w+1)\Gamma_k(\alpha n+\alpha\nu)}{\Gamma_k(\alpha n+\alpha+\nu+kw+k)} && (56)
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t u^w {}_{k,j}\mathbb{E}_{t-u}^{\alpha,\gamma}(\nu,\lambda) du &= k^{\frac{\nu}{k}+w+1}\Gamma(w+1) t^{\frac{\alpha+\nu}{k}+\frac{w+k}{k}-1} k^{-w} && (57)\\
&\times \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{\Gamma_k(\alpha n+\alpha+\nu+kw+k)(n+j)!} && (58)\\
&= \Gamma_k(kw+k) k^{\frac{\nu+k}{k}} t^{\frac{\alpha+nu+w+k}{k}-1} && (59)\\
&\times \sum_{n=0}^{\infty} \frac{(\gamma)_{n+j,k}\left(\lambda t^{\frac{\alpha}{k}}\right)^n}{\Gamma_k(\alpha n+\alpha+\nu+kw+k)(n+j)!} && (60)\\
&= \Gamma_k(kw+k)_{k,j}\mathbb{E}_t^{\alpha,\gamma}(\gamma+kw+k,\lambda) && (61)
\end{aligned}$$

If $k=1$, we have

$$\int_0^t u^w E_{t-u}(\nu,\lambda) du = \Gamma(w+1) E_t(\nu+w+1,\lambda) \tag{62}$$

that coincides with formula in page 69 of [4].

Conclusion

We have introduce a generalization of the classical Miller-Ross function doing it by rising the k-Gamma function and the Pochhammer k-symbol.

Elementary properties were demonstrated as well as know particular cases were obtained.

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