

A NON-ISOTHERMAL CONSOLIDATION MODEL FOR GRADIENT-BASED POROPLASTICITY

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Abstract. In this work, the thermodynamically consistent non-local model for concretes subjected to high temperatures originally proposed by Ripani, et al (2014) [1] is extended, in order to evaluate the failure behavior of partially saturated quasi brittle materials like soils. This new formulation follows the gradient-based poroplastic theory proposed by the authors [2], moreover, introduces the temperature as an additional variable of the internal characteristic length. The non-local effect is achieved assuming that the internal variables are the only ones of non-local character. Hence, both q and q_α will be considered as arguments in the free energy. On the other hand, in order to reproduce the softening behavior of partially saturated soils the saturation degree as well as the confinement level should be considered in the internal characteristic length calibration. Finally, the FE implementation of this enriched constitutive theory with selective C1 and C0 interpolation functions for the internal variables and the kinematic fields, respectively, is discussed.

1 INTRODUCTION

The mechanics of porous media constitutes a discipline of great relevance in several knowledge areas like Geophysics, Biomechanics and Materials Science. Its main aim is

the description of the kinematic and pore pressure of porous continua when subjected to arbitrary mechanical and/or physical actions.

The definitive advantages of porous mechanics to macroscopically describe or predict complex response behavior of cohesive-frictional materials based on fundamental aspects of their microstructure while accounting for the hydraulic properties and their influence in the resulting failure mechanism were recognized by several authors in the scientific community [3, 9]. The influence of the non-isothermal condition in the failure behaviour was also evaluate by [22, 17]. Consequently, a tendency to replace the theoretical framework of classical continuum mechanics with that of non-linear porous mechanics was observed. Firstly this task took place in case of soil mechanics, see a.o. [7, 5], and subsequently in the field of concrete, see a.o. [27, 19] and, furthermore, of biomaterials , see a.o. [16, 20].

Further development in classical continuum mechanics was the extension to non-local concepts. The main aim was the regularization of post peak response behavior regarding mesh size and element orientation in case of finite element analyses, based on fundamental aspects of the material microstructure, see a.o. [25, 1, 32].

In recent years significant progresses and relevant contributions were made in non-local gradient formulations for non-porous materials. Thermodynamic frameworks were considered in the proposals of [1, 21, 28, 8, 32]. Considerations of material anisotropy in the formulation of internal variables evolution laws in case of gradient plasticity are due to [2, 29]. Formulation of gradient enhanced coupled damage-plasticity material models and related finite element implementations, see [25, 12, 6].

Recently, non-local concepts were extended for the formulation of porous material models, see a.o. [11, 10, 15, 14]. In spite of the strong development of constitutive modelling for porous media, explained before, there is still a need of thermodynamically consistent theoretical frameworks. This is particularly the case of non-local models for porous materials. Thermodynamic concepts should lead to dissipative stress formulations in hardening and softening regimes that allow non-constant descriptions of the internal variables of non-local character to accurately predict the sensitivity of porous material failure behavior to both confinement and saturation levels.

In this work, the thermodynamically consistent formulation for non-porous gradient-based elastoplasticity by Vrech and Etse [32] which follows general thermodynamic approach proposed by Svedberg and Runesson [25] for non-local damage formulation is extended for porous media. Main feature of present proposal is the definition of a gradient-based characteristic length in terms of both the governing stress and hydraulic conditions to capture the variation of the transition from brittle to ductile failure mode of cohesive-frictional porous materials with the confinement level and saturation [14].

2 Thermodynamics of Porous Continua

The postulate of local state stipulates that the internal energy of a homogeneous system is independent of the evolution rate and it can be characterized by the same state variables as the ones characterizing equilibrium states. The postulate of local state is extended to

porous continua by considering that their thermodynamics are obtained by adding the thermodynamic contributions of each constituent, that are the solid skeleton and the fluid continuum.

2.1 First law of the Thermodynamics

The first law of Thermodynamics expresses the conservation of energy in all forms. The energy $\dot{\mathbb{E}}$ of a system can be expressed as the sum of its kinetic $\dot{\mathcal{K}}$ and internal energy $\dot{\mathcal{E}}$ of each component of this system. Considering a body occupying the volume Ω , with boundary $\partial\Omega$, the first law of the thermodynamic is expressed as

$$\dot{\mathbb{E}} = \dot{\mathcal{K}} + \dot{\mathcal{E}} = \mathcal{P}_{ext} + \mathcal{Q} \quad (1)$$

with

$$\begin{aligned} \dot{\mathcal{E}} &= \frac{d}{dt} \int_{\Omega} e \, d\Omega & ; \dot{\mathcal{K}} &= \frac{1}{2} \int_{\Omega} \rho^s (1 - \phi) |\dot{u}_i \dot{u}_i| + \rho^f \phi |w_i w_i| \, d\Omega \\ \mathcal{Q} &= \int_{\Omega} \rho r \, d\Omega - \int_{\partial\Omega} h_i n_i \, d\partial\Omega & ; \mathcal{P}_{ext} &= \int_{\Omega} \rho b_i \dot{u}_i \, d\Omega + \int_{\partial\Omega} \sigma_{ij} n_i \dot{u}_j - \frac{p}{\rho^f} w_i n_i \, d\partial\Omega \end{aligned} \quad (2)$$

Here, internal energy density (per unit mass) e is the internal energy density (per unit mass), body force b_i is the body force, total stress σ_{ij} is the stress, pore pressure p is the pore pressure, heat source density r is a heat source density and heat flux h_i is the heat flux. The displacement u_i , the unit normal vector on $\partial\Omega$, n_i , and the mass density ρ , were also included.

Considering the equilibrium equation, the explicit form of the internal energy density for local dissipative porous material follows from Eq.(1) as

$$\rho \dot{e} = \sigma_{ij} \dot{\epsilon}_{ij} - h_f M_{i,i} - h_{i,i} + \rho r \quad (3)$$

being ϕ the porosity, h_f the fluid specific enthalpy and M_i the fluid flow vector.

2.2 The Second law of Thermodynamics

While, the first law states the conservation of energy in all of its forms, the second law states that the energy can only deteriorate. The second law introduces a new physical quantity, the entropy, which can only increase when an isolate system is considered. Let entropy volume density (per unit mass) s be an entropy volume density (per unit mass), and according to the second law of thermodynamic the entropy $\dot{\mathcal{S}}$ of a thermodynamic system can not decrease. Thus

$$\dot{\mathcal{S}} + \mathcal{Q}_\theta \geq 0 \quad (4)$$

with

$$\dot{S} = \int_{\Omega} \rho \dot{s} + (s_f M_i)_{,i} \, d\Omega \quad ; \quad \mathcal{Q}_{\theta} = \int_{\Omega} \frac{\rho r}{T} \, d\Omega - \int_{\partial\Omega} \frac{n_i h_i}{T} \, d\partial\Omega \quad (5)$$

being \mathcal{Q}_{θ} the entropy flux and T the temperature. Transforming the surface integral of Eq. (5) into a volume integral, it follows that the volume integral in Eq. (4) must be non-negative for any system Ω , which yields

$$\rho \dot{s} + (s_f M_i)_{,i} + \left(\frac{h_i}{T} \right)_{,i} - \frac{\rho r}{T} \geq 0 \quad (6)$$

Then, invoking the first law by Eq. (3) and introducing the Helmholtz's free energy $\Psi = e - sT$ as well as the free enthalpy of the fluid per mass unit (or Gibbs potential) $g_f = h_f - s_f T$, the following expression is attained

$$\sigma_{ij} \dot{\epsilon}_{ij} - g_f M_{i,i} - s \dot{T} - \dot{\Psi} - M_i (s_f T_{,i} + g_{f,i}) - \frac{h_i}{T} T_{,i} \geq 0 \quad (7)$$

Finally, considering the mass balance equation, $\dot{m} + M_{i,i} = 0$ the Eq. (7) can be rewritten in the form

$$\Phi_s + \Phi_f + \Phi_T \geq 0 \quad (8)$$

with

$$\Phi_s = \sigma_{ij} \dot{\epsilon}_{ij} + g_f \dot{m} - s \dot{T} - \dot{\Psi} \quad (9)$$

$$\Phi_f = -M_i (s_f T_{,i} + g_{f,i}) \quad (10)$$

$$\Phi_T = -\frac{h_i}{T} T_{,i} \quad (11)$$

The first component of Eq. (8) is related to the skeleton dissipation, Φ_s , the second accounts for the viscous dissipation due to the relative motion of the fluid with respect to the skeleton and the last source of dissipation, Φ_T , involves the temperature gradient $T_{,i}$ and therefore is related to the dissipation due to heat conduction.

Owing to the additive character of the Helmholtz free energy and entropy, $\Psi = \Psi_s + m\Psi_f$ and $s = s_s + m s_f$, the fluid state equations and the relation $m = \rho^f \phi$ allow to express Φ_s as

$$\Phi_s = \sigma_{ij} \dot{\epsilon}_{ij} + p\dot{\phi} - s_s \dot{T} - \dot{\Psi}_s \quad (12)$$

This expression of the skeleton dissipation Φ_s matches the standard expression of the dissipation of a solid phase. Indeed, the strain work rate of an ordinary solid would reduce to the term $\sigma_{ij} \dot{\epsilon}_{ij}$. In the case of a porous continuous, the strain work rate related to the skeleton is obtained by adding $p\dot{\phi}$, to account for the action of the pore pressure on the skeleton through the internal walls of the porous network.

3 Thermodynamically consistent gradient-based thermo-poro-plastic theory

The thermodynamic framework of classical or local plasticity is extended to non-local gradient-based elastoplastic porous material subjected to high temperatures. Following [23, 4] we assume that arbitrary thermodynamic states of the dissipative material during isothermal processes are completely determined by the elastic strain $\varepsilon_{ij}^e = \varepsilon_{ij} - \varepsilon_{ij}^p$ or the temperature T , the elastic entropy $s^e = s - s^p$ and the internal variables q_α with $\alpha = s, p$ for solid or porous phase, which are considered here as scalar variables.

When considering poroplastic materials the elastic porosity $\phi^e = \phi - \phi^p$ needs also to be included as a thermodynamic argument, see [4]. Based on [25, 32, 14] we further assume that the internal variables q_α , are the only ones of non-local character. Hence, both q_α and $q_{\alpha,i}$ will appear as arguments in the free energy Ψ_s , such that

$$\Psi_s = \Psi_s(\varepsilon_{ij}^e, \phi^e, T, q_\alpha, q_{\alpha,i}) \quad (13)$$

Upon differentiation of Eq. (13) and combining with the intrinsic dissipation of Eq. (7) on the whole domain Ω , integrating the gradient term by parts and applying the Divergence Theorem, it follows

$$\int_{\Omega} \left[\left(\sigma_{ij} - \frac{\partial \Psi_s}{\partial \varepsilon_{ij}^e} \right) \dot{\varepsilon}_{ij} + \left(p - \frac{\partial \Psi_s}{\partial \phi^e} \right) \dot{\phi} + \left(-s_s - \frac{\partial \Psi_s}{\partial T} \right) \dot{T} + \frac{\partial \Psi_s}{\partial \varepsilon_{ij}^e} \dot{\varepsilon}_{ij}^p + \frac{\partial \Psi_s}{\partial \phi^e} \dot{\phi}^p + \sum_{\alpha} Q_{\alpha} \dot{q}_{\alpha} \right] d\Omega + \int_{\partial\Omega} \sum_{\alpha} Q_{\alpha}^{(b)} \dot{q}_{\alpha} d\partial\Omega \geq 0 \quad (14)$$

The stress dissipation on the boundary $\partial\Omega$ is defined as $Q_{\alpha}^{(b)} = -\partial \Psi_s / \partial q_{\alpha,i} n_i$, also, in the domain Ω , the dissipative stress Q_{α} can be decomposed into the local and non-local components, local dissipative stress Q_{α}^{loc} and non-local dissipative stress Q_{α}^{nloc} , respectively

$$Q_{\alpha} = Q_{\alpha}^{loc} + Q_{\alpha}^{nloc} \quad (15)$$

with

$$Q_{\alpha}^{loc} = -\frac{\partial \Psi_s}{\partial q_{\alpha}} \quad ; \quad Q_{\alpha}^{nloc} = -\left(\frac{\partial \Psi_s}{\partial q_{\alpha,i}} \right)_{,i} \quad (16)$$

In standard form (as for local theory), it is postulated that the last inequality must hold for any choice of domain Ω and for any independent thermodynamic process. As a result, Coleman's equation are formally obtained like for the local continuum theory.

$$\sigma_{ij} = \frac{\partial \Psi_s}{\partial \varepsilon_{ij}^e} \quad ; \quad p = \frac{\partial \Psi_s}{\partial \phi^e} \quad ; \quad s_s = -\frac{\partial \Psi_s}{\partial T} \quad (17)$$

being the dissipative energy

$$\mathfrak{D} = \sigma_{ij}\dot{\varepsilon}_{ij}^p + p\dot{\phi}^p + \sum_{\alpha} Q_{\alpha}\dot{q}_{\alpha} \geq 0 \quad \text{in } \Omega \quad (18)$$

$$\mathfrak{D}^{(b)} = \sum_{\alpha} Q_{\alpha}^{(b)}\dot{q}_{\alpha} \geq 0 \quad \text{on } \partial\Omega \quad (19)$$

In the particular case of non-porous material ($p = 0$) above equations take similar forms to those obtained by [25, 32] for isothermal situations.

From the above Eq. (18) and Eq. (19), it can be concluded that the difference between this simplified non-local theory and the local one is the additional gradient term in the expression of the dissipative stresses Q_{α} , and the boundary dissipation term dissipative stresses on the boundary $Q_{\alpha}^{(b)}$.

3.1 Thermodynamically consistent constitutive relations

Based on previous works [25, 32], the following additive expression of the free energy corresponding to non-local gradient poroplastic materials is adopted

$$\Psi_s(\varepsilon_{ij}^e, m^e, T, q_{\alpha}, q_{\alpha,i}) = \Psi^e(\varepsilon_{ij}^e, m^e, T) + \Psi^{p,loc}(q_{\alpha}) + \Psi^{p,nloc}(q_{\alpha,i}) \quad (20)$$

whereby Ψ^e is the elastic energy of non-isothermal porous media deduced in [22, 4].

$$\Psi^e = \frac{1}{2}\varepsilon_{ij}^e C_{ijkl} \varepsilon_{kl}^e + \frac{1}{2}M(\phi^e)^2 - MB_{ij}\varepsilon_{ij}^e \phi^e - \frac{1}{2}\chi T^2 + \ell \rho_f \phi^e T - A_{ij}\varepsilon_{ij}^e T \quad (21)$$

Whereas local plasticity contributions to the free energy $\Psi^{p,loc}$ and non-local contributions to the free energy $\Psi^{p,nloc}$ are the local and non-local gradient contributions due to dissipative hardening/softening behaviors, which are expressed in terms of the internal variables q_{α} and their gradient $q_{\alpha,i}$, respectively.

Once the Coleman's relationships are deduced from Eq. (17) the following expressions can be obtained

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^e - MB_{ij}\phi^e - A_{ij}T \quad (22)$$

$$p = -MB_{ij}\varepsilon_{ij}^e + M\phi^e + \ell \rho_f T \quad (23)$$

$$s_s = A_{ij}\varepsilon_{ij}^e - \ell \rho_f \phi^e + \chi T \quad (24)$$

being M the Biot's module [13], $B_{ij} = b\delta_{ij}$ with Biot coefficient b the Biot coefficient, and $C_{ijkl} = C_{ijkl}^s + MB_{ij}B_{kl}$ is the undrained elastic constitutive tensor, whereby C_{ijkl}^s is the fourth-order elastic tensor which linearly relates stress and strain. Also, χ is the porous media heat capacity, ℓ is the latent heat of variation in fluid mass content and $A_{ij} = \alpha_{\theta}\delta_{ij}$ is the thermal expansion tensor, with α_{θ} the thermal expansion coefficient.

3.2 Rate form of constitutive equations

Considering the additive decomposition of the free energy potential in Eq. (20) and the flow rule, the following rate expressions are obtained from Eqs. 22-24

$$\dot{\sigma}_{ij} = C_{ijkl}^s \dot{\varepsilon}_{kl} - B_{ij} \dot{p} + (B_{ij} \ell \rho_f - A_{ij}) \dot{T} - C_{ijkl}^s \dot{\lambda} \frac{\partial g}{\partial \sigma_{kl}} \quad (25)$$

$$\dot{p} = -M B_{ij} \dot{\varepsilon}_{ij} + M \dot{\phi} + \ell \rho_f \dot{T} + M B_{ij} \dot{\lambda} \frac{\partial g}{\partial \sigma_{ij}} - M \dot{\lambda} \frac{\partial g}{\partial p} \quad (26)$$

$$\dot{s}_s = A_{ij} \dot{\varepsilon}_{ij} - \ell \rho_f \dot{\phi} + \chi \dot{T} - A_{ij} \dot{\lambda} \frac{\partial g}{\partial \sigma_{kl}} + \ell \rho_f \dot{\lambda} \frac{\partial g}{\partial p} \quad (27)$$

while the evolution law of the local and non-local dissipative stress in Eq. (15) results

$$\dot{Q}_\alpha = \dot{Q}_\alpha^{loc} + \dot{Q}_\alpha^{nloc} \quad (28)$$

with

$$\dot{Q}_\alpha^{loc} = -\dot{\lambda} H_\alpha^{loc} \frac{\partial g}{\partial Q_\alpha} \quad (29)$$

$$\dot{Q}_\alpha^{nloc} = l_\alpha^2 \left(H_{\alpha ij}^{nloc} \dot{\lambda}_{,j} \frac{\partial g}{\partial Q_\alpha} + \dot{\lambda} H_{\alpha ij}^{nloc} Q_{\alpha,j} \frac{\partial^2 g}{\partial Q_\alpha^2} \right)_{,i} \quad (30)$$

Thereby, local hardening/softening module local hardening/softening module H_α^{loc} have been introduced as well as the new non-local hardening/softening tensor non-local hardening/softening module $H_{\alpha ij}^{nloc}$ as defined in [25]

$$H_\alpha^{loc} = \frac{\partial^2 \Psi^{p,loc}}{\partial q_\alpha^2}, \quad H_{\alpha ij}^{nloc} = \frac{1}{l_\alpha^2} \frac{\partial^2 \Psi^{p,nloc}}{\partial q_{\alpha,i} \partial q_{\alpha,j}} \quad (31)$$

$H_{\alpha ij}^{nloc}$ is a second order positive defined tensor. The internal characteristic length, l_α , is a physical entity that characterizes the material microstructure [18, 24, 30].

4 A finite element formulation for gradient-based thermo-poro-plasticity

Having established the basic principles of the thermodynamically consistent gradient-based theory for non-isothermal porous media the present section focuses in the formulation of a new C_1 -continuous FE formulation in order to solve the boundary value problem with the capacity to reproduce both localized and diffuse failure modes that characterized quasi-brittle materials like concretes and soils, see Mroginski and Etse [13], Ripani et al [22] and Coussy [4]

4.1 Incremental formulation

An incremental formulation of the above boundary value problem introduces residual terms, what makes the stress update necessary. The transition from elastic to plastic regimes within a loading step must also be considered. At the end of the $j + 1$ iteration of current load step, the incremental equilibrium condition, the fluid mass balance, the heat conservation and the yield condition are studied in a weak form. Thereby, bold symbol for tensors are used instead of the indicial notation employed in previous sections.

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T : \boldsymbol{\sigma}_{j+1} \, d\Omega - \int_{\partial\Omega} \delta \mathbf{u}^T \mathbf{t}_{j+1} \, d\partial\Omega = 0 \quad (32)$$

$$\int_{\Omega} \delta p \dot{m}_{j+1} \, d\Omega - \int_{\Omega} \nabla \delta p \cdot \mathbf{w}_{j+1} \, d\Omega + \int_{\partial\Omega} \delta p \mathbf{w}_{j+1} \cdot \mathbf{n} \, d\partial\Omega = 0 \quad (33)$$

$$\int_{\Omega} \delta T \dot{s}_{s_{j+1}} \, d\Omega - \int_{\Omega} \frac{\mathbf{q}_{j+1}}{T_0} \nabla \delta T \, d\Omega = + \int_{\partial\Omega} \delta T \frac{\mathbf{q}_{j+1}}{T_0} \cdot \mathbf{n} \, d\partial\Omega = 0 \quad (34)$$

$$\int_{\Omega} \delta \lambda f(\boldsymbol{\sigma}, p, Q_{\alpha})|_{j+1} \, d\Omega = 0 \quad (35)$$

In contrast to the local plasticity algorithm, Eq. (35) is not strictly satisfied but in a weak form. Furthermore, it is only fulfilled when the convergence is reached and not necessarily during the iterative process.

Considering the decomposition of the stress tensor in the $j + 1$ iteration as $\boldsymbol{\sigma}_{j+1} = \boldsymbol{\sigma}_j + \Delta \boldsymbol{\sigma}$, and replacing in Eq. (32) it results

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T : \Delta \boldsymbol{\sigma} \, d\Omega = \int_{\partial\Omega} \delta \mathbf{u}^T \mathbf{t}_{j+1} \, d\partial\Omega - \int_{\Omega} \delta \boldsymbol{\varepsilon}^T : \boldsymbol{\sigma}_j \, d\Omega \quad (36)$$

Then, the replacement of $\Delta \boldsymbol{\sigma}$ in the last equation by the linearized form of Eq. (25), results in

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}^T : (\mathbf{C}^s : \Delta \boldsymbol{\varepsilon} - \mathbf{B} \Delta p + (\mathbf{B} \ell \rho_f - \mathbf{A}) \Delta T - \mathbf{C}^s : \mathbf{g}^s \Delta \lambda) \, d\Omega = \int_{\partial\Omega} \delta \mathbf{u}^T \mathbf{t}_{j+1} \, d\partial\Omega - \int_{\Omega} \delta \boldsymbol{\varepsilon}^T : \boldsymbol{\sigma}_j \, d\Omega \quad (37)$$

It can be observed that Eq. (37) is very similar to the incremental equilibrium condition of classical plasticity as it does not include an explicit dependence on the Laplacian of the plastic multiplier.

Considering the incremental decomposition of the infiltration vector $\mathbf{w}_{j+1} = \mathbf{w}_j + \Delta \mathbf{w}_{j+1}$ and the rate of the fluid mass content \dot{m} , as well as the generalized Darcy's law for porous media [4], $\mathbf{w} = -\mathbf{k} \cdot \nabla p$, the Eq. (33) can be reformulated as

$$\int_{\Omega} \delta p \left(\frac{\Delta p}{M} + \mathbf{B} : \Delta \boldsymbol{\varepsilon} - \frac{\ell \rho_f}{M} \Delta T - (\mathbf{B} : \mathbf{g}^s - g^p) \Delta \lambda \right) d\Omega =$$

$$- \Delta t \int_{\Omega} \nabla \delta p \cdot \mathbf{k} \cdot \nabla p_j d\Omega - \Delta t \int_{\Omega} \nabla \delta p \cdot \mathbf{k} \cdot \nabla \Delta p d\Omega - \Delta t \int_{\partial\Omega} \delta p \mathbf{w}_{j+1} \cdot \mathbf{n} d\partial\Omega \quad (38)$$

Then, considering Fourier law $\mathbf{q}_{j+1} = -\kappa \nabla T$ and the entropy density rate \dot{s}_s in Eq. (27), the weak form of the heat conservation law in Eq. (34), can be reformulated as

$$\int_{\Omega} \delta T \left[(\mathbf{A} - \ell \rho_f \mathbf{B}) \Delta \boldsymbol{\varepsilon} - \frac{\ell \rho_f}{M} \Delta p + \left(\frac{(\ell \rho_f)^2}{M} + \chi \right) \Delta T + (\ell \rho_f \mathbf{B} - \mathbf{A}) \mathbf{g}^s \Delta \lambda \right] d\Omega =$$

$$- \Delta t \int_{\Omega} \frac{\kappa}{T_0} \nabla \delta T \cdot \nabla \Delta T d\Omega - \Delta t \int_{\Omega} \frac{\kappa}{T_0} \nabla \delta T \cdot \nabla T_j d\Omega - \Delta t \int_{\partial\Omega} \delta T \frac{\mathbf{q}_{j+1}}{T_0} \cdot \mathbf{n} d\partial\Omega \quad (39)$$

Following [18], the yield function f can be approximated with sufficient accuracy by means of a linear Taylor series around $(\boldsymbol{\sigma}_j, p_j, T_j, Q_{\alpha_j})$,

$$f(\boldsymbol{\sigma}, p, T, Q_{\alpha})|_{j+1} = f(\boldsymbol{\sigma}, p, T, Q_{\alpha})|_j + \mathbf{f}^s : \Delta \boldsymbol{\sigma} + f^p \Delta p + f^T \Delta T + f_{\alpha}^Q \Delta Q_{\alpha} \quad (40)$$

Also, from the additive decomposition of the dissipative stress in Eq. (28) it follows

$$\dot{Q}_{\alpha} = \dot{Q}_{\alpha}^{loc} + \dot{Q}_{\alpha}^{nloc} = -H_{\alpha}^{loc} g_{\alpha}^Q \dot{\lambda} + l_{\alpha}^2 \mathbf{H}_{\alpha}^{nloc} g_{\alpha}^Q \nabla^2 \dot{\lambda} \quad (41)$$

By replacing Eq. (25) and Eq. (41) into Eq. (40) the weak form of the yield condition is obtained

$$\int_{\Omega} \delta \lambda f(\boldsymbol{\sigma}, p, Q_{\alpha})|_{j+1} d\Omega = \int_{\Omega} \delta \lambda f(\boldsymbol{\sigma}, p, Q_{\alpha})|_j d\Omega + \int_{\Omega} \delta \lambda \mathbf{f}^s : \mathbf{C}^s : \Delta \boldsymbol{\varepsilon} d\Omega$$

$$+ \int_{\Omega} \delta \lambda (f^p - \mathbf{f}^s : \mathbf{B}) \Delta p d\Omega + \int_{\Omega} \delta \lambda (f^T + \ell \rho_f \mathbf{B} - \mathbf{A}) \Delta T d\Omega - \int_{\Omega} \delta \lambda \mathbf{f}^s : \mathbf{C}^s : \mathbf{g}^s \Delta \lambda d\Omega$$

$$+ \int_{\Omega} \delta \lambda f_{\alpha}^Q (-H_{\alpha}^{loc} g_{\alpha}^Q \Delta \lambda + l_{\alpha}^2 \mathbf{H}_{\alpha}^{nloc} g_{\alpha}^Q \nabla^2 \Delta \lambda) d\Omega = 0 \quad (42)$$

4.2 Galerkin discretization

As it can be observed in Eqs. (37) - (39) and (Eq. (42)) at most first order derivatives of the displacement, pore pressure and temperature fields appear as well as second order derivative of the plastic multiplier. Therefore, displacement, pressure and temperature

fields discretizations require C_0 -continuous shape functions that are indicated as \mathbf{N}_u , \mathbf{N}_p and \mathbf{N}_T , respectively. However, C_1 -continuous shape functions, called \mathbf{H} , are required for the plastic multiplier discretization. Then, the Finite Element approximations can be expressed as

$$\mathbf{u} = \mathbf{N}_u \bar{\mathbf{u}} \quad ; \quad p = \mathbf{N}_p \bar{p} \quad ; \quad T = \mathbf{N}_T \bar{T} \quad ; \quad \lambda = \mathbf{H} \bar{\lambda} \quad (43)$$

where $\bar{\mathbf{u}}$, \bar{p} , \bar{T} and $\bar{\lambda}$ are the nodal displacement vector, the pore pressure, the nodal temperature and the plastic multiplier, respectively. Hence considering $\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \nabla^s \mathbf{N}_u \bar{\mathbf{u}} = \bar{\mathbf{B}} \bar{\mathbf{u}}$ and replacing the above entities in Eqs. (37) - (39) and (Eq. (42)) the following set of integral equations is obtained

$$\begin{aligned} & \left\{ \int_{\Omega} \delta \bar{\mathbf{u}}^T \bar{\mathbf{B}}^T : \mathbf{C}^s : \bar{\mathbf{B}} \, d\Omega \right\} \Delta \bar{\mathbf{u}} - \left\{ \int_{\Omega} \delta \bar{\mathbf{u}}^T \bar{\mathbf{B}}^T : \mathbf{B} \mathbf{N}_p \, d\Omega \right\} \Delta \bar{p} \\ & + \left\{ \int_{\Omega} \delta \bar{\mathbf{u}}^T \bar{\mathbf{B}}^T : (\mathbf{B} \ell \rho_f - \mathbf{A}) \mathbf{N}_T \, d\Omega \right\} \Delta \bar{T} - \left\{ \int_{\Omega} \delta \bar{\mathbf{u}}^T \bar{\mathbf{B}}^T : \mathbf{C}^s : \mathbf{g}^s \mathbf{H} \, d\Omega \right\} \Delta \bar{\lambda} = \\ & \int_{\partial\Omega} \delta \bar{\mathbf{u}}^T \mathbf{N}_u^T \mathbf{t}_{j+1} \, d\partial\Omega - \int_{\Omega} \delta \bar{\mathbf{u}}^T \bar{\mathbf{B}}^T : \boldsymbol{\sigma}_j \, d\Omega \quad (44) \end{aligned}$$

$$\begin{aligned} & \left\{ \int_{\Omega} \delta \bar{p} \mathbf{N}_p^T \mathbf{B} : \bar{\mathbf{B}} \, d\Omega \right\} \Delta \bar{\mathbf{u}} + \left\{ \int_{\Omega} \delta \bar{p} \left[\frac{\mathbf{N}_p^T \mathbf{N}_p}{M} + \Delta t (\nabla \mathbf{N}_p)^T \cdot \mathbf{k} \cdot \nabla \mathbf{N}_p \right] \, d\Omega \right\} \Delta \bar{p} \\ & - \left\{ \int_{\Omega} \delta \bar{p} \mathbf{N}_p^T \frac{\ell \rho_f}{M} \mathbf{N}_T \, d\Omega \right\} \Delta \bar{T} + \left\{ \int_{\Omega} \delta \bar{p} \mathbf{N}_p^T [\mathbf{g}^p - \mathbf{B} : \mathbf{g}^s] \mathbf{H} \, d\Omega \right\} \Delta \bar{\lambda} = \\ & - \left\{ \Delta t \int_{\Omega} \delta \bar{p} (\nabla \mathbf{N}_p)^T \cdot \mathbf{k} \cdot \nabla \mathbf{N}_p \, d\Omega \right\} \bar{p}_j - \Delta t \int_{\partial\Omega} \delta \bar{p} \mathbf{N}_p^T \mathbf{w}_{j+1} \cdot \mathbf{n} \, d\partial\Omega \quad (45) \end{aligned}$$

$$\begin{aligned} & \left\{ \int_{\Omega} \delta \bar{T} \mathbf{N}_T^T (\mathbf{A} - \ell \rho_f \mathbf{B}) : \bar{\mathbf{B}} \, d\Omega \right\} \Delta \bar{\mathbf{u}} - \left\{ \int_{\Omega} \delta \bar{T} \mathbf{N}_T^T \frac{\ell \rho_f}{M} \mathbf{N}_p \, d\Omega \right\} \Delta \bar{p} \\ & + \left\{ \int_{\Omega} \delta \bar{T} \left[\mathbf{N}_T^T \left(\frac{(\ell \rho_f)^2}{M} + \chi \right) \mathbf{N}_T + \Delta t \frac{\kappa}{T_0} (\nabla \mathbf{N}_T)^T \cdot \nabla \mathbf{N}_T \right] \, d\Omega \right\} \Delta \bar{T} \\ & + \left\{ \int_{\Omega} \delta \bar{T} \mathbf{N}_T^T (\ell \rho_f \mathbf{B} - \mathbf{A}) : \mathbf{g}^s \mathbf{H} \, d\Omega \right\} \Delta \bar{\lambda} = \\ & - \left\{ \Delta t \int_{\Omega} \delta \bar{T} \frac{\kappa}{T_0} (\nabla \mathbf{N}_T)^T \cdot \nabla \mathbf{N}_T \, d\Omega \right\} \bar{T}_j - \Delta t \int_{\partial\Omega} \delta \bar{T} \mathbf{N}_T^T \frac{\mathbf{q}_{j+1}}{T_0} \cdot \mathbf{n} \, d\partial\Omega \quad (46) \end{aligned}$$

$$\begin{aligned}
 & \left\{ \int_{\Omega} \delta \bar{\lambda} \mathbf{H}^T \mathbf{f}^s : \mathbf{C}^s : \bar{\mathbf{B}} \, d\Omega \right\} \Delta \bar{\mathbf{u}} + \left\{ \int_{\Omega} \delta \bar{\lambda} \mathbf{H}^T [\mathbf{f}^p - \mathbf{f}^s : \mathbf{B}] \mathbf{N}_p \, d\Omega \right\} \Delta \bar{p} \\
 & \quad + \left\{ \int_{\Omega} \delta \bar{\lambda} \mathbf{H}^T [\mathbf{f}^T + \ell \rho_f \mathbf{B} - \mathbf{A}] \mathbf{N}_T \, d\Omega \right\} \Delta \bar{T} \\
 & \quad + \left\{ - \int_{\Omega} \delta \bar{\lambda} \mathbf{H}^T [\mathbf{f}^s : \mathbf{C}^s : \mathbf{g}^s + \bar{H}_{\alpha}^{loc}] \mathbf{H} + l_{\alpha}^2 \mathbf{H}^T \bar{\mathbf{H}}_{\alpha}^{nloc} \mathbf{P} \, d\Omega \right\} \Delta \bar{\lambda} = \\
 & \quad - \int_{\Omega} \delta \bar{\lambda} \mathbf{H}^T f(\boldsymbol{\sigma}_j, p_j, Q_{\alpha_j}) \, d\Omega \quad (47)
 \end{aligned}$$

where

$$\nabla^2(\Delta \lambda) = \nabla^2(\mathbf{H}) \Delta \bar{\lambda} = \mathbf{P} \Delta \bar{\lambda} \quad (48)$$

$$\bar{H}_{\alpha}^{loc} = \mathbf{f}_{\alpha}^Q H_{\alpha}^{loc} \mathbf{g}_{\alpha}^Q \quad (49)$$

$$\bar{\mathbf{H}}_{\alpha}^{nloc} = \mathbf{f}_{\alpha}^Q \mathbf{H}_{\alpha}^{nloc} \mathbf{g}_{\alpha}^Q \quad (50)$$

Equations (44)-(47) must hold for any admissible variation of $\delta \bar{\mathbf{u}}$, $\delta \bar{p}$, $\delta \bar{T}$ and $\delta \bar{\lambda}$. Thus, the algebraic equation system in matrix form of the proposed FE formulation for gradient-dependent thermoporoplastic media can be expressed as

$$\begin{bmatrix}
 -\mathbf{K}_{ss} & \mathbf{Q}_{sp} & \mathbf{Q}_{sT} & \mathbf{Q}_{s\lambda} \\
 \mathbf{Q}_{ps} & \mathbf{K}_{pp} + \Delta t \mathbf{H}_{pp} & \mathbf{Q}_{pT} & \mathbf{Q}_{p\lambda} \\
 \mathbf{Q}_{Ts} & \mathbf{Q}_{Tp} & \mathbf{K}_{TT} + \Delta t \mathbf{H}_{TT} & \mathbf{Q}_{T\lambda} \\
 \mathbf{Q}_{\lambda s} & \mathbf{Q}_{\lambda p} & \mathbf{Q}_{\lambda T} & -\mathbf{K}_{\lambda\lambda}
 \end{bmatrix}
 \begin{bmatrix}
 \Delta \bar{\mathbf{u}} \\
 \Delta \bar{p} \\
 \Delta \bar{T} \\
 \Delta \bar{\lambda}
 \end{bmatrix}
 =
 \begin{bmatrix}
 \mathbf{F}_s^{\text{int}} - \mathbf{F}_s^{\text{ext}} \\
 -\mathbf{F}_p \\
 -\mathbf{F}_T \\
 -\mathbf{F}_{\lambda}
 \end{bmatrix} \quad (51)$$

Submatrices of Eq. (51) were obtained by inspection from Eqs. (44)-(47). Also, in Table 1 the solution algorithm of the boundary value problem is summarized.

The main and most important difference between this selective C_1 -continuous FE formulation and the one based on C_0 continuity approximations for gradient plasticity proposed by [26, 31] is the solution procedure. While present formulation requires only the solution of Eq. (51), the FE approaches proposed by the aforementioned authors require an additional global iteration to obtain the plastic multiplier.

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<p>1) Compute matrices of Eq. (51)</p> <p>2) Solve the algebraic system of Eq. (51) in terms of the increments $\Delta\bar{\mathbf{u}}$, $\Delta\bar{p}$, $\Delta\bar{T}$ and $\Delta\bar{\lambda}$</p> <p>3) Update primary variables $\Delta\bar{\mathbf{u}}_{j+1} = \Delta\bar{\mathbf{u}}_j + \Delta\bar{\mathbf{u}}$, $\Delta\bar{p}_{j+1} = \Delta\bar{p}_j + \Delta\bar{p}$, $\Delta\bar{T}_{j+1} = \Delta\bar{T}_j + \Delta\bar{T}$ and $\Delta\bar{\lambda}_{j+1} = \Delta\bar{\lambda}_j + \Delta\bar{\lambda}$</p> <p>4) On each integration point compute:</p> $\Delta\boldsymbol{\varepsilon}_{j+1} = \bar{\mathbf{B}} \Delta\bar{\mathbf{u}}_{j+1}$ $\Delta\lambda_{j+1} = \mathbf{H} \Delta\bar{\lambda}_{j+1}$ $\nabla^2(\Delta\lambda_{j+1}) = \mathbf{P} \Delta\bar{\lambda}_{j+1}$ $q_{\alpha_{j+1}} = q_{\alpha_0} + g_{\alpha}^Q \Delta\lambda_{j+1}$ $\nabla^2 q_{\alpha_{j+1}} = \nabla^2 q_{\alpha_0} + g_{\alpha}^Q \nabla^2(\Delta\lambda_{j+1})$ $\boldsymbol{\sigma}^t = \boldsymbol{\sigma}_0 + \mathbf{C}^s : \Delta\boldsymbol{\varepsilon}_{j+1} - \mathbf{B}\mathbf{N}_p \Delta\bar{p}_{j+1}$ <p>IF $f(\boldsymbol{\sigma}^t, q_{\alpha}, \nabla^2 q_{\alpha}) _{j+1} > 0$</p> $\boldsymbol{\sigma}_{j+1} = \boldsymbol{\sigma}^t - \Delta\lambda_{j+1} \mathbf{C}^s : \mathbf{g}^s$ <p>ELSE</p> $\boldsymbol{\sigma}_{j+1} = \boldsymbol{\sigma}^t$ <p>END</p> <p>5) Check convergence criterion, i.e. balance between internal and external energy. If it is not achieved go to 1</p>
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Table 1: Gradient-thermo-poro-plasticity algorithm for C_1 -continuous FE

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